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APPLIED MECHANICS

APPLIED MECHANICS

By

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PREFACE TO SECOND EDITION

The general arrangement of subject matter in the second edition of Applied Mechanics follows closely the order so favorably received by the students and teachers who have used the first edition. The author has attempted to produce a book which will please the student.

The entire book has been reset. Where classroom experience has indicated that a part needed amplification, that part has been rewritten; and many additional illustrative examples have been added to help clarify the principles presented. In the sections dealing with kinematics the notation has been brought in line with current practice and the treatment has been expanded. The discussions dealing with kinetics also have been considerably amplified, especially where variable forces are involved. However, it is believed that there is no excess wordage and that the student will obtain a sound understanding of the basic principles without becoming lost in unnecessary details.

There has been some rearrangement of problem material. The data of many problems have been changed, and about three hundred new problems have been added. The book now contains over one thousand examples and problems. Much time and energy have been devoted to the selection of these problems. The student who solves a reasonable number of them will be well prepared for his dependent courses.

The author is greatly indebted to his former students and to many teachers who have used the first edition for valuable suggestions. Wherever possible these suggestions have been incorporated into the second edition. He also acknowledges his indebtedness to his colleagues at Purdue University, because any book such as this must necessarily represent a melting down of the ideas and experiences which are born from a close association of many years.

H. F. Girvin.

Purdue University
May, 1949

PREFACE TO FIRST EDITION

For many years Applied Mechanics has generally been considered a subject which should be taught during the junior year of the various engineering courses after the student had completed the required work in Physics and Calculus.

During the last four or five years many colleges have rearranged their engineering curricula. At most institutions this has resulted in Applied Mechanics becoming more or less of a sophomore subject. This change has placed additional responsibility on the teacher in the form of larger classes of less mature students.

The experienced teacher readily observes a distinct difference in the ability of a sophomore and a junior to do the type of analytical thinking which is required of students of Applied Mechanics. Furthermore a junior is more apt to have had some contact with industry or construction work in a practical way. This seems to give some students a marked advantage in the early part of the work at least. In writing this text the author has tried to produce a book which will be easily understood by these less mature students, yet which does not do all the work for the student. For the student will benefit from a course in Applied Mechanics only in proportion to what he puts into it.

The material covered is more than can be studied during the time allowed for the usual required course. However, the arrangement is such that the text will be found to be readily adjustable to the requirements of the individual instructor.

More space has been devoted to graphical methods than is ordinarily found in books of this type, but the graphical solutions have been collected into separate chapters. These chapters may be omitted entirely, if time is not available for their consideration, without affecting the continuity of the mathematical solutions. With some additional problem material by the instructor these chapters could easily be used as a text for a course in Graphical Statics.

Other special features are the introduction of separate chapters on Kinematics of a Particle and Kinematics of a Rigid Body. Also, the work on Loaded Cables, Plane Motion, and Impact has been carried a little farther than is customary. In the Statics part of the book all examples and problems are stated with the

data in numerical form; but in the portion of the book devoted to Dynamics quite a number of examples and problems have the data expressed algebraically. This is done so that the instructor may introduce the principle of the dimensional check if he so desires.

The book contains many more problems than can be covered in the time usually allowed for Applied Mechanics. These problems are arranged according to their difficulty. Some of the problems will test the ability of the most capable students.

The author wishes to express his obligations to his colleagues at Purdue University, especially Prof. A. P. Poorman whose excellent books he has used in his classes for a number of years and which have helped greatly to formulate his ideas on the subject. He is also indebted to Dean A. A. Potter of the Schools of Engineering and Dean R. G. Dukes, Head of the Applied Mechanics Department at Purdue for their kind encouragement during the preparation of the book. He also wishes to thank Mr. Arthur E. Koenig and Mrs. Girvin for assistance in reading the manuscript and proofs.

H. F. GIRVIN.

Purdue University
April 9, 1938.

CONTENTS

APPLIED MECHANICS

CHAPTER 1

FUNDAMENTAL CONCEPTIONS

1. Definitions.—Mechanics is the science which treats of the effect of forces on the form, motion, and general behavior of matter, whether gaseous, fluid, or solid. Thus is stated the broad or general definition of mechanics.

Applied Mechanics or Engineering Mechanics is gradually increasing its field. This expansion of interest is clearly demonstrated by examining the titles of the papers published by the Applied Mechanics Division of the American Society of Mechanical Engineers. In the catalogs of engineering colleges we find such courses as Hydraulics, Fluid Mechanics, Strength and Elasticity of Materials, Applied Mechanics, Theory of Elasticity, and other allied subjects grouped under the general head of Engineering Mechanics.

Hydraulics and Fluid Mechanics deal with the mathematical theory involved in the behavior of fluids (liquids and gases) under static and mobile conditions.

Strength of Materials or Resistance of Materials deals with the theory involved in the computation of the internal stresses which are set up in the various parts of an engineering structure. Such theory may be used in determining the size and shape of new construction or in the investigation of parts or structures already in service.

Under the title of Theory of Elasticity the more advanced problems of design are usually grouped.

Specifically, Applied Mechanics is the title generally given to the study of the effect of forces on particles and rigid bodies. A particle is a body or portion of a body the dimensions of which are insignificant in terms of its surroundings and its motion. A rigid body is any quantity of matter the particles of which do not move relative to each other. The condition of internal stress and distortion of bodies due to the action of the forces is disregarded. The subject of Applied Mechanics is divided into two parts, Statics and Dynamics.

(*a*) *Statics* is the study of the effect of forces on bodies at rest or in a state of uniform motion.

(*b*) *Dynamics* is the study of particles and physical bodies in motion. It is subdivided into Kinematics and Kinetics.

Kinematics is the study of motion without reference to the forces which cause or influence the motion.

Kinetics is the study of the effect of unbalanced forces on the motion of bodies which therefore have accelerated or non-uniform motion.

This book will deal exclusively with Statics, Kinematics, and Kinetics.

2. **Fundamental Quantities.**—In the study of Mechanics certain fundamental quantities, such as *force*, *distance*, *time*, and *mass*, are involved. These quantities are measured by comparison with certain standards, which have been selected or determined by qualified committees or by recognized authorities.

A force is commonly thought of as a push or pull, exerted by one body on another. A force may be defined as the effect of one body on another in changing or tending to change the state of motion of the body acted upon. This effect may be demonstrated by either or both of the following: (1) change of motion or of the resistance to motion of the opposing body; (2) change of shape of the resisting body.

A man may pull on a weight so that it slides along a level surface. He is exerting a force on the weight; and the weight is pulling on the man, at the same time, with a force that is of equal magnitude but is opposite in direction. The weight, resting on the plane surface, exerts a downward pressure on the surface due to the pull of gravity; and the surface exerts an equal and opposite upward pressure on the weight. These examples are illustrations of what Sir Isaac Newton called his third law: *For every action there is an equal, opposite, and collinear reaction.* It is therefore impossible to have a single force acting; there must always be an equal and opposite force. Forces always occur in pairs.

In the English-speaking countries the foot-pound-second system of units is in general use. In the United States the pound force is equal to the pull which the earth exerts on a certain mass known as the "Standard Pound." This mass is preserved in the United States Bureau of Standards.

Forces are sometimes classified according to their method of application. A concentrated force is one which may be considered

as acting at a given point. A distributed force is one which acts over an area, as water pressure against a dam, earth pressure against a retaining wall, or the pressure on the head of any pressure-containing vessel.

Distances are measured in feet or inches. These units of measure are also based on certain standards which are preserved in the Bureau of Standards.

The unit of time which is most commonly used is the second, although in some cases the larger units of minutes and hours are used.

Mass is quantity of matter or anything which occupies space. The unit of mass is the amount which will receive an acceleration of one foot per second per second when acted upon by a force of one pound.

In his second law Newton states: *A body acted on by a resultant force receives an acceleration which is directly proportional to the force, and inversely proportional to the mass of the body.* Thus,

$$F=Ma=\frac{Wa}{g}$$

in which F = force acting on a body, in pounds;
M = mass of the body;
a = acceleration of the body, in feet per second per second;
W = weight of body, in pounds;
g = acceleration of gravity, in feet per second per second.

From this equation it is seen that, if $g=32.2$ ft per sec per sec and a 1-lb force is to produce an acceleration of 1 ft per sec per sec, W must be 32.2 lb. The unit of mass is therefore taken as 32.2 pounds of matter.

3. **Vectors.**—Quantities which possess magnitude only, such as areas, volumes, and masses, are scalar quantities.

Quantities which involve direction as well as magnitude, such as velocities, accelerations, and forces, are vector quantities.

In the solution of problems, graphical methods are sometimes used. Although many students think that graphical methods are not accurate, the degree of accuracy depends almost entirely on the amount of care used in executing the drawings. That is, with reasonable skill in the use of drawing instruments and with draw-

ings of proper size, solutions by the graphical method will produce results at least as accurate as those obtained with the slide rule.

In applying the graphical method we represent forces by vectors. In order that a vector may represent a force, the vector must have *magnitude*, or a definite length according to some scale; it must have *direction*; and it must have a definite *position* in a definite plane.

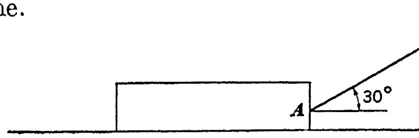


FIG. 1

In Fig. 1, P represents a 100-lb force, to a scale 100 lb to the inch, acting at a point A and pulling up to the right at an angle of 30° to the horizontal.

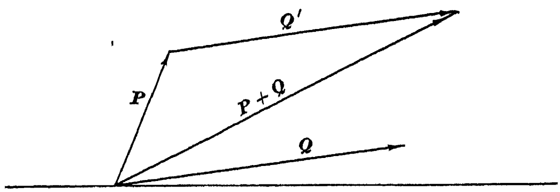


FIG. 2

Vectors may be added or subtracted. In Fig. 2, P and Q are two vectors which are to be added. From the head of P lay off Q' equal and parallel to Q; then vector $P+Q$ is the sum of vectors P and Q.

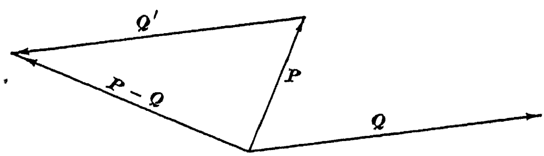


FIG. 3

Vector subtraction is the addition of a negative quantity to a positive quantity. In Fig. 3, from the head of vector P lay off Q' equal and parallel to Q, but with its direction reversed. The vector $P-Q$ is then the vector difference of vectors P and Q.

4. **Transmissibility of Forces.**—The point of application of a force may be moved along the line of action of the force without

changing the external effect of the force on the body. If a block is placed on a smooth plane surface, as in Fig. 4, and a force P applied at a point A as shown, the block will slide. Next, if a hole, as indicated by the dotted lines, is bored in the block and the point of application is moved to the point B, the external effect of the force on the block will be unchanged.

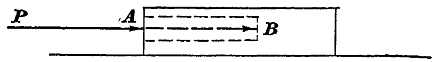


FIG. 4

5. Moment of a Force.—Experience teaches us that, when we apply a wrench to a pipe, the longer the handle of the wrench the greater the turning effect produced; and also that the pull should be applied approximately at right angles to the handle to produce the greatest turning effect.

The turning moment of a force, or—as more commonly expressed—the moment of a force, is the measure of the turning effect produced by the force. The moment of a force is the product of the force and the perpendicular distance from the line of action of the force to the axis of rotation. In Fig. 5 the moment of the force F with respect to the axis OY is the product of the force F and distance r; thus, $M_O = Fr$.

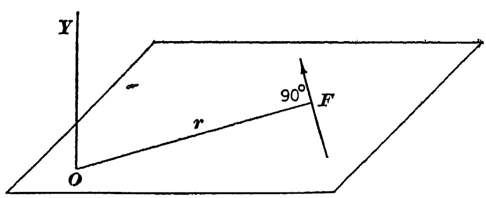


FIG. 5

Most texts on Mechanics call moments which produce counterclockwise rotation positive and those which produce clockwise rotation negative.

6. Free-Body Diagram.—The free-body diagram is a device which has been evolved as an aid to the solution of problems in Mechanics. It enables the student to get a better conception of what forces are acting and how they are acting on the body under consideration. Fundamentally, the idea of the free-body diagram is to show the body or a particular part of it isolated from all

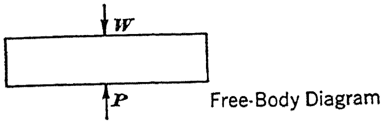


FIG. 6

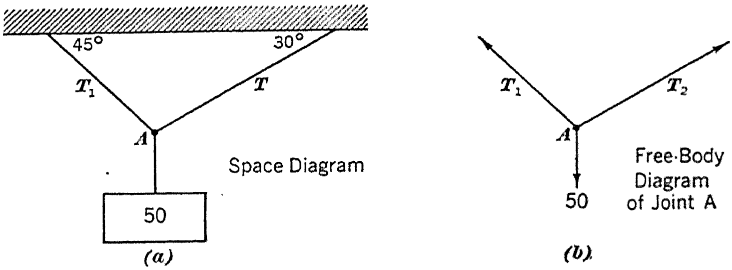


FIG. 7

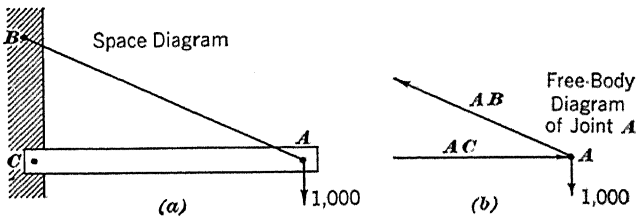


FIG. 8

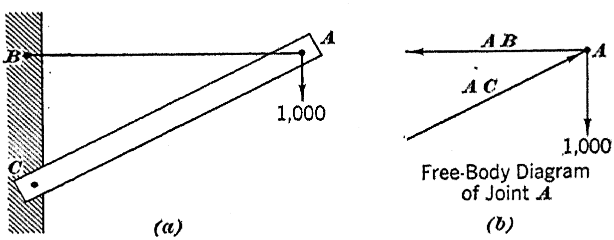


FIG. 9

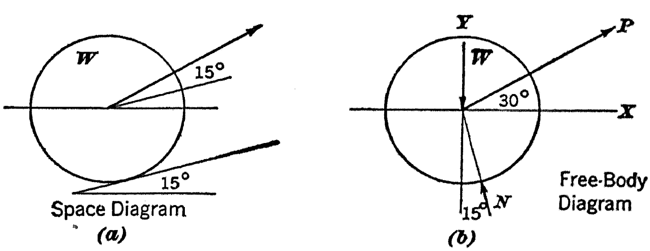


FIG. 10

physical contact with any other body, or other parts of the same body, and yet to have it remain in its original position with reference to all other bodies. As an example, let a book rest on a table, and then remove the table but have the book retain its original position with reference to the floor and all other objects. In order that this may be possible, the upward push of the table on the book must be supplied by an assumed force P. The free-body diagram for the book is shown in Fig. 6. Several other free-body diagrams are shown in Figs. 7, 8, 9, and 10.

In Fig. 7 (*a*) a weight is shown supported by two cords. The point of intersection of the cords is the free body in this case, as it is the object on which the forces are acting. Fig. 7 (*b*) is the free-body diagram. In Figs. 8 and 9 the pin at point A is the free body or object on which the forces are acting. In each case the pin is shown isolated from everything else *but held in position by the forces exerted on it by the other parts of the structure.* Fig. 10 (*a*) represents a roller on an inclined plane; Fig. 10 (*b*) shows the roller as a free body.

7. **Method of Solution of Problems.**—In general the method of procedure in solving a problem is as follows:

(*a*) Draw a space diagram or a diagrammatic sketch, showing all dimensions and external forces acting on the body.

(*b*) Draw a free-body diagram, showing all known and unknown forces acting on the body or part of the body being studied.

(*c*) Solve for the unknown forces by one or more of the following methods:

1. Graphical solution.
2. Trigonometric solution.
3. Algebraic solution.
4. Solution by moments.

8. **Dimensional Equations.**—In solving problems, the various terms of the equations represent physical quantities. The units in which these terms are expressed must be so selected that the equations will be homogeneous. The mathematical significance of this statement is that the quantities entering into the several terms of the equation must be expressed in such units that, when a dimensional equation is written, each term of the equation will reduce to the same units.

As an example, consider the equation $F=\frac{Wa}{g}$, which is the mathematical statement of Newton's second law given in Art. 2. If this equation is expressed in dimensional form, it becomes:

$$\text{pounds}=\frac{\text{pounds}\times\text{feet}}{\frac{\text{feet}}{(\text{seconds})^2}\times(\text{seconds})^2}$$

or

$$\text{pounds}=\text{pounds}$$

The equation is thus dimensionally correct. Consider also the equation $v^2=v_o^2+2gs$. This is one of the fundamental equations of rectilinear motion. If this equation is expressed in dimensional form, it becomes:

$$\left(\frac{\text{feet}}{\text{seconds}}\right)^2=\left(\frac{\text{feet}}{\text{seconds}}\right)^2+\frac{2\text{ feet}}{(\text{seconds})^2}\times\text{feet}$$

If the above equation is multiplied by $(\text{seconds})^2$, the resulting equation is:

$$(\text{feet})^2=(\text{feet})^2+2(\text{feet})^2$$

9. **Equilibrium.**—By definition, equilibrium is a balanced condition. From the standpoint of Mechanics, a body is in equilibrium when at rest or moving in a straight line with a constant velocity.

This condition of equilibrium is one of the principal working tools of the science of Statics. The idea will also be used in this book as a means of making the solution of problems in Kinetics more readily understood.

A book resting on a plane surface is an example of static equilibrium. The book is acted upon by a balanced force system, consisting of the downward pull of gravity or the weight of the book and the equal and opposite upward push of the plane surface. Every object which is at rest is thus necessarily being subjected to the action of a balanced force system. In the same way any object which is moving in a straight line at a constant velocity is being acted upon by a balanced force system. If it were not, it would receive an acceleration, which would be proportional to the resultant force acting and inversely proportional to the mass of the body.

CHAPTER 2

COPLANAR, CONCURRENT FORCE SYSTEMS

10. Definitions.—A coplanar, concurrent force system is a group of forces all of which lie in the same plane, as in the plane of the paper, and which also intersect in a common point.

The resultant of any system of forces is the minimum system of forces which will produce the same effect as the original system. Such a minimum system may be: (1) a single force; (2) a pair of equal, opposite, and parallel forces, which in Mechanics is designated as a *couple* (see Art. 28); (3) a single force and a couple (see Art. 29).

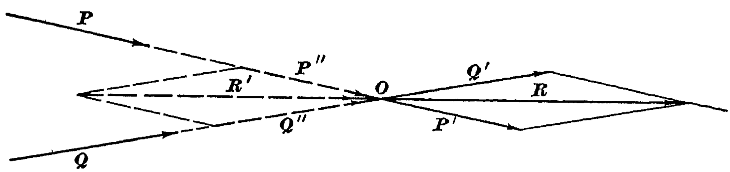


FIG. 11

The equilibrant of any system of forces is: (1) a single force; (2) a pair of equal, opposite, and parallel forces (a couple); or (3) a single force and a couple. In any case the equilibrant has the same magnitude as the resultant of the system, but is of opposite sense. The equilibrant is that which will cancel the resultant.

The components of a force are the two or more forces which, acting together, will produce the same effect as the original force acting alone.

11. Resultant of Two Forces by Graphical Method.—In Fig. 11, P and Q are two forces the resultant of which is to be found. Extend the lines of action of P and Q until they meet at point O (Art. 4). From O lay off vector P' equal to P and vector Q' equal to Q; complete the parallelogram. The diagonal R is the resultant of P and Q. The same result may be obtained by placing the heads of vectors P'' and Q'' at point O and completing the parallelogram as indicated by the dotted lines. *The essential point*

of this construction is that either the two heads or the two tails of the two vectors must be at the point O. This construction, which is known as the Parallelogram Law, is the work of Simon Stevinus (1548–1620).

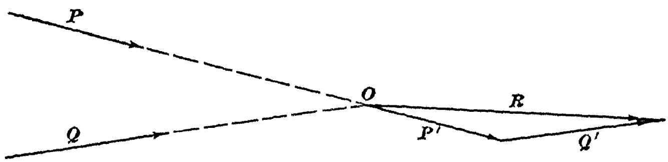


FIG. 12

The resultant of two forces may also be determined by the method known as the Triangle Law. Produce the lines of action of the two forces until they intersect at point O, Fig. 12. Starting at point O lay off vector P', to scale, equal to P; and from the head of P' lay off Q' equal and parallel to Q. The vector R is the sum of P' and Q', or of P and Q. The resultant R and the two component forces must pass through the point O.

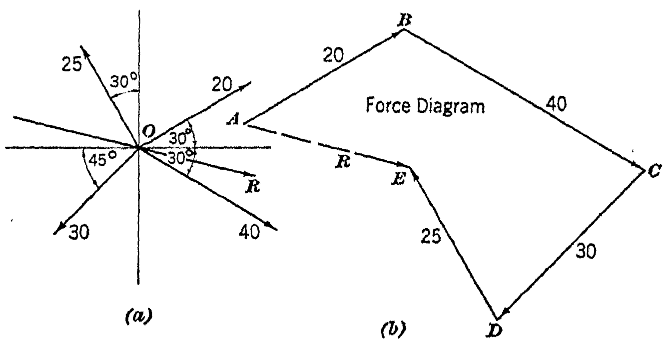


FIG. 13

12. Resultant of Three or More Forces Graphically.—Either the Parallelogram Law or the Triangle Law may be extended, so that the resultant of three or more forces may be determined. The resultant of any two forces may be found, and this resultant may be combined with a third force to find a second resultant. This procedure may be continued until any number of forces have been reduced to a single resultant force. This resultant force would pass through the intersection of the several forces.

When the resultant of several forces is desired, the Triangle Law is the better method, as the construction is more easily made. To determine the resultant of the forces in Fig. 13 (*a*), proceed as follows: Starting at point *A* in Fig. 13 (*b*), lay off *AB* to scale equal and parallel to the 20-lb force of Fig. 13 (*a*). From *B* lay off *BC* equal and parallel to the 40-lb force, *CD* equal and parallel to the 30-lb force, and *DE* equal and parallel to the 25-lb force. Connect *A* and *E*; then *AE* is the resultant of the force system, in amount and direction. Since the resultant and the component forces must pass through a common point, a line drawn through *O* in Fig. 13 (*a*) parallel to *AE* will be the line of action of the resultant of the system. If the construction of Fig. 13 (*b*) is studied, it will be observed that it is simply an extension of the Triangle Law or vector addition.

PROBLEMS

1. Determine the amount and direction of the resultant of a 100-lb force acting to the right at 15° above the horizontal and a 200-lb force acting to the right at 60° above the horizontal. Use both the parallelogram method and the triangle method. *Ans. 280 lb; 45.33°.*

2. Find the amount and direction of the resultant of the forces shown in Fig. 14.

3. Find the amount of each of the two rope pulls in Fig. 15.

4. Reverse the direction of the 200-lb force in Fig. 14. Determine the resultant of this system of forces.

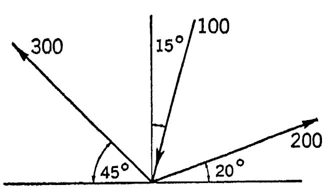


FIG. 14

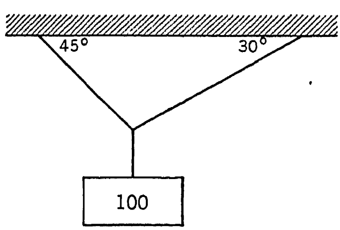


FIG. 15

13. Components of a Force.—In many cases it is desirable to resolve a force into its components. Graphically, this is the reverse of the Triangle Law construction. In Fig. 16 (*a*) the force *P* is resolved into three components *A*, *B*, and *C*. In Fig. 16 (*b*), force *P* is resolved into two components *D* and *E*. These are components because by vector addition $A \mathrel{+\!\!\!\rightarrow} B \mathrel{+\!\!\!\rightarrow} C = P$ and $D \mathrel{+\!\!\!\rightarrow} E = P$. It is thus seen that a force may be resolved into any

number of components. The construction of Fig. 16 (*a*) or Fig. 16 (*b*) gives the amount and direction of the components only. As stated in Art. 12, the components and the resultant force must pass through a common point. This point may be any point on the line of action of the resultant force.

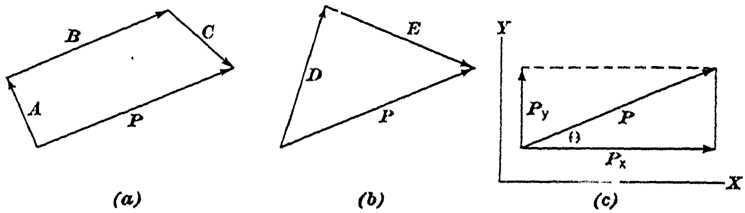


FIG. 16

The components which are most often desired are those parallel to some set of rectangular axes. In Fig. 16 (*c*) the force P is resolved into components parallel to the X and Y axes. From the end of vector P, drop perpendiculars to the X and Y axes; then P_x is the component parallel to the X axis and P_y is the component parallel to the Y axis. Thus,

$$P_x = P \cos\theta \text{ and } P_y = P \sin\theta$$

PROBLEMS

5. A force of 500 lb acts to the right at 30° above the horizontal. Determine the horizontal and vertical components. *Ans. H, 433 lb; V, 250 lb.*

6. Resolve the force of Problem 5 into components, making angles of 45° and 75° with the positive end of the X axis.

7. A force of 300 lb acts to the left at an angle of 60° above the horizontal. Resolve this force into rectangular components, one of which makes an angle of 60° with the positive end of the X axis.

8. A force of 4,000 lb acts up to the right at an angle of 60° with the X axis. Resolve this force into components acting at 105° and 345° with the X axis.

9. A force of 8,000 lb at an angle of 30° with the X axis was resolved into two components, the magnitudes of which were 6,000 lb and 10,000 lb. Determine the direction of each component.

14. Calculation of the Resultant of Two Forces.—Since the force polygon for two forces and their resultant is a triangle, Fig. 17 (*b*), the amount and direction of the resultant can be determined by solving this triangle by the usual trigonometric formulas. The

angle θ between the forces P and Q is the supplement of the angle opposite R. The angles α and β can be found by the sine law.

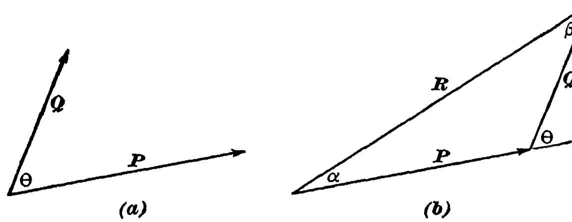


FIG. 17

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \alpha} = \frac{R}{\sin (180° - \theta)}$$

$$\tan \alpha = \frac{Q \sin \theta}{P + Q \cos \theta}$$

PROBLEMS

10. Why is the sign of the last term in the above equation for R^2 positive rather than negative, as the cosine law is usually written?

11. Determine the magnitude and direction of the resultant of a 100-lb force acting to the right at 15° above the horizontal and a 200-lb force acting to the right at 75° above the horizontal. *Ans. 265 lb; 55.86°.*

12. A 50-lb force acts horizontally to the right and a 100-lb force acts to the left at 30° above the horizontal. Determine the magnitude and direction of the resultant.

13. Determine the magnitude and the direction of the resultant of the 400-lb and 300-lb forces in Fig. 18 (a).

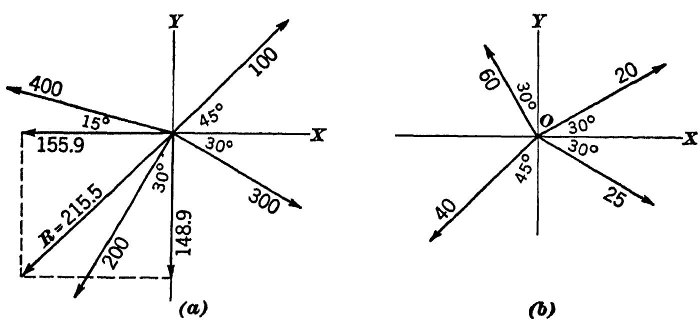


FIG. 18

14. Determine the magnitude and direction of the resultant of the 20-lb and 40-lb forces of Fig. 18 (b).

15. Calculation of the Resultant of Three or More Forces. The method of Art. 14 may be extended to cover the calculation of the resultant of three or more forces, but this becomes a rather involved procedure.

If each force in turn is resolved into its X and Y components, the X components may be added algebraically, and the Y components added algebraically. The original force system is then reduced to a force ΣF_x along the X axis and a force ΣF_y along the Y axis. The resultant of these two forces is obtained from the equation

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

The angle which this resultant R makes with the X axis is $\tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$. The resultant passes through the point of intersection of the original forces.

EXAMPLE

Determine the amount and direction of the resultant of the force system shown in Fig. 18 (a).

Each force is resolved into its X and Y components. These components, with their proper algebraic signs, are added.

$$\Sigma F_x = 70.7 - 100.0 + 259.8 - 386.4 = -155.9 \text{ lb} \leftarrow$$
$$\Sigma F_y = 70.7 - 173.2 - 150.0 + 103.6 = -148.9 \text{ lb} \downarrow$$
$$R = \sqrt{155.9^2 + 148.9^2} = 215.5; \tan\theta = \frac{148.9}{155.9} = .955; \theta = 223.7°$$

PROBLEMS

15. Determine the amount and direction of the resultant of the force system shown in Fig. 18 (b). *Ans. 28.7 lb; 132.4°.*

16. In Fig. 18 (a) reverse the direction of the 100-lb and 400-lb forces and compute the amount and direction of the resultant. Check by graphics.

17. In Fig. 18 (b) reverse the direction of the 25-lb and 60-lb forces and compute the resultant.

16. Varignon's Theorem or the Principle of Moments.—A very important theorem of Mechanics is as follows: *The moment of a resultant force, with respect to any axis perpendicular to the plane of the resultant force, is equal to the algebraic sum of the moments of the component forces with respect to the same axis.*

In Fig. 19 is given Varignon's proof for the case of two coplanar concurrent forces. For other systems the theorem will be considered axiomatic. In the illustration R is the resultant of forces S and T, as is shown by the parallelogram construction. Let B be the trace of the axis, in the plane of the forces, with respect to which moments are to be taken. Draw a line through A and B, and drop perpendiculars on AB from the ends of S and R to the points C and D; also drop a perpendicular on FD from the end of S to E; and from point B drop perpendiculars s, r, and t to forces S, R, and T. Then,

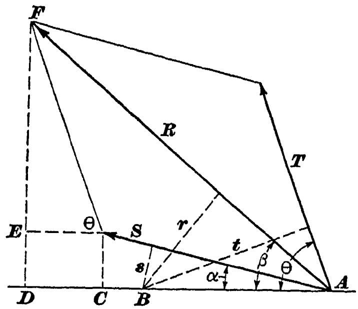


FIG. 19

$$R \sin \beta = DE + EF$$

$$R \sin \beta = S \sin \alpha + T \sin \theta$$

$$R \sin \beta \times AB = S \sin \alpha \times AB + T \sin \theta \times AB$$

$$Rr = Ss + Tt$$

Since S and T may be the resultants of other forces, the principle just proved may be extended as follows: *If a force is broken up at any point on its line of action into any number of coplanar components, the algebraic sum of the moments of the component forces with respect to any axis perpendicular to their plane is equal to the moment of the resultant with respect to that axis.*

PROBLEM

18. In Fig. 18 (b), take a point on the X axis 10 in. to the right of point O. Compute the moment of the resultant force with respect to an axis through this point and perpendicular to the plane of the paper. Check the result by finding the algebraic sum of the moments of the component forces with respect to this axis (lay down to scale and get distances graphically).

17. Conditions for Equilibrium of a Coplanar, Concurrent Force System.—As stated in Art. 9 equilibrium means a balanced condition. A body in equilibrium is being acted upon by a balanced force system. The body either is at rest or is moving with a constant velocity.

If a body is being acted upon by a coplanar concurrent force system, there can be no rotation, since all forces pass through a common point. All that is necessary to produce a balanced condition is that the sum of the vertical components of the forces equal zero, or $\Sigma F_v = 0$; and that the sum of the horizontal components of the forces equal zero, or $\Sigma F_h = 0$. The same thing may be stated in a more general way: The algebraic sum of the components of the forces along each of any two intersecting straight lines must be zero. This means that there can be no resultant force acting along either of the two intersecting lines, and therefore the resultant of the force system is zero; thus, $R = 0$.

If the resultant is zero, it follows from the principle of moments, Art. 16, that the algebraic sum of the moments of the component forces with respect to any axis perpendicular to the plane of the forces must be zero; that is, $\Sigma M = 0$.

From this discussion it is evident that two independent equations can be written in the form $\Sigma F_x = 0$ and $\Sigma F_y = 0$. These two equations can be solved simultaneously for two unknowns. Therefore, a coplanar concurrent force system can have only two unknown quantities, if a definite solution is to be made. These unknown quantities may be:

1. The amount and direction of one force;
2. The magnitudes of two forces;
3. The directions of two forces;
4. The magnitude of one force, and the direction of another force.

Since $R = 0$, it follows that if all the forces are added vectorially, as in Art. 12, the force polygon will be a closed figure. It is therefore evident that graphically the condition necessary to establish equilibrium of a coplanar, concurrent force system is that $R = 0$, or that the force polygon close. The force polygon cannot be closed to give a definite solution if there are more than two unknowns.

For coplanar, concurrent force systems, the conditions which must be satisfied if equilibrium is to exist can be summarized as:

1. Graphically, $R = 0$, or the force polygon must close;
2. Algebraically, $R = 0$, or $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

18. Solution of Problems.—Several problems will now be solved by each of several different methods. It is desirable that

the student know how to apply each of these methods. They are the essential working tools of the science of Statics. Certain types of problems are more readily solved by one method than by another.

By carrying these methods along in parallel, the advantages of one over the other will soon be observed. It is essential that the student learn to carry out the solution of problems in a systematic manner, drawing the free-body diagrams carefully and learning to break the problem up into its elementary parts.

Ability to analyze and to carry on a systematic process of attack is one of the most important benefits to be obtained by the study of Mechanics.

EXAMPLE 1

A 1,000-lb weight is suspended by means of three ropes meeting at A, Fig. 20 (a). Determine the tension in the ropes AB and AC.

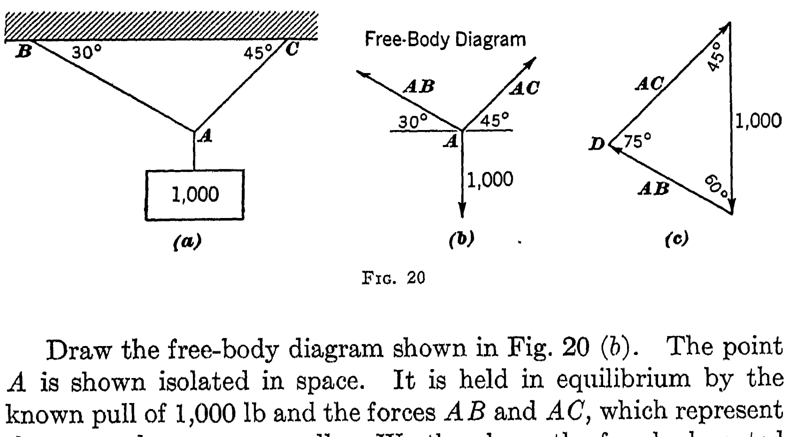


FIG. 20

Draw the free-body diagram shown in Fig. 20 (b). The point A is shown isolated in space. It is held in equilibrium by the known pull of 1,000 lb and the forces AB and AC, which represent the two unknown rope pulls. We thus have the free body acted upon by one known force and two unknown forces.

Graphical Solution.—Draw to scale a vector to represent the known 1,000 lb force, Fig. 20 (c). Through the lower end of this vector draw a line parallel to force AB, Fig. 20 (b), and through the upper end draw a line parallel to force AC. These lines intersect at point D, forming a closed force triangle. Notice that the force arrows follow around the triangle as in vector addition. The vectors AB and AC represent the tensions in the two ropes to the scale that was used in laying off the 1,000-lb force.

Trigonometric Solution.—By the sine law from the force triangle of Fig. 20 (*c*),

$$\frac{1{,}000}{\sin 75^\circ}=\frac{AB}{\sin 45^\circ}=\frac{AC}{\sin 60^\circ}$$

or

$$\frac{1{,}000}{0.965}=\frac{AB}{0.707}=\frac{AC}{0.866}$$

$$AB=731 \text{ lb, T. and } AC=896 \text{ lb, T.}$$

Algebraic Method.—In the free-body diagram of Fig. 20 (*b*), sum the horizontal components of all forces and equate to zero. Sum the vertical components of all forces and equate to zero. Each of these equations contains the same two unknown quantities; therefore they may be solved simultaneously.

$$\begin{array}{ll}
\Sigma F_x=0 & \Sigma F_y=0 \\
-AB\cos 30^\circ+AC\cos 45^\circ=0 & AB\sin 30^\circ+AC\sin 45^\circ \\
-0.866\,AB+0.707\,AC \qquad =0 & \qquad -1{,}000=0 \\
-0.5\;\;AB+0.707\,AC-1{,}000=0 & 0.5\,AB+0.707\,AC-1{,}000=0 \\
\hline
-1.366\,AB \qquad\qquad +1{,}000=0 &
\end{array}$$

$$AB=731 \text{ lb, T.}$$
$$AC=896 \text{ lb, T.}$$

In solving problems by the algebraic method, it is possible by proper choice of axes to eliminate one of the unknown quantities from each equation, and thus avoid the necessity of solving simultaneous equations. This may be done by summing forces along a line which is perpendicular to the unknown to be eliminated. *A force has no component along any line which is perpendicular to the line of action of the original force.* Therefore, a force produces no effect in a direction at right angles to its line of action.[1]

[1] It is suggested that the student study the above statement very carefully. He should form a clear picture of the physical facts involved. He must see that, if a force acts normally to a plane or line, the force has no component parallel to the surface or line in question.

If an object rests on a smooth horizontal plane, with no forces acting on it but its weight, or the pull of gravity down, and the upward reaction of the plane, why does it remain at rest? If it is placed on a smooth inclined plane, why does it slide down the plane? In the first case there is no force acting parallel to the plane to produce motion. In the second case the weight of the object has a component parallel to the plane. It is this component which causes the object to slide down the plane.

It should thus be evident that, if the axis of summation is chosen so that it is perpendicular to a given force, the given force will have no component parallel to the axis of summation. This makes it possible to write a summation equation for any force system so as to eliminate from the equation any particular force.

In Fig. 21 (a), the X axis is taken perpendicular to force AC. The force AC therefore has no component along this line.

$$\Sigma F_x = 0$$

$$AB \cos 15° - 1{,}000 \cos 45° = 0$$

$$AB = \frac{1{,}000 \times 0.707}{0.966} = 731 \text{ lb, T.}$$

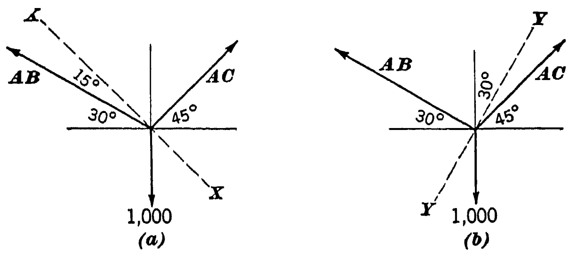


Fig. 21

In Fig. 21 (b), the Y axis is taken perpendicular to force AB. The force AB has no component along this line.

$$\Sigma F_y = 0$$

$$AC \cos 15° - 1{,}000 \cos 30° = 0$$

$$AC = \frac{1{,}000 \times 0.866}{0.966} = 896 \text{ lb, T.}$$

EXAMPLE 2

Determine the stresses in members AB and AC of the crane shown in Fig. 22 (a).

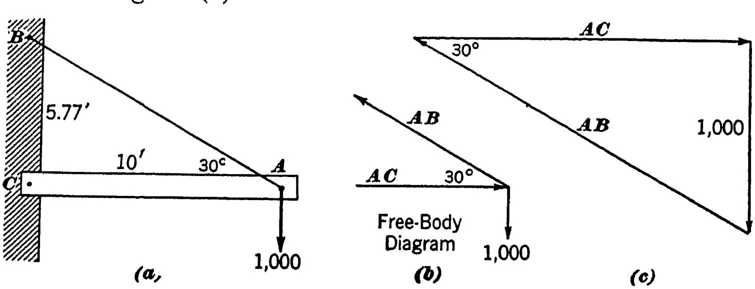


Fig. 22

Graphical Solution.—Fig. 22 (b) is the free-body diagram for the pin at A. The pin is held in equilibrium by the known force

of 1,000 lb, the unknown tension in AB, and the unknown compression in AC.

To draw the force triangle of Fig. 22 (c), lay off the 1,000-lb vector to scale; through the lower end draw a line parallel to AB; and through the upper end draw a line parallel to AC. The vectors AB and AC of Fig. 22 (c) represent the unknown stresses to the scale that was used in laying down the 1,000-lb force.

Trigonometric Solution.—

$$\frac{1{,}000}{\sin 30^\circ}=\frac{AB}{\sin 90^\circ}=\frac{AC}{\sin 60^\circ}$$

$$AB=2{,}000 \text{ lb, T. and } AC=1{,}730 \text{ lb, C.}$$

It will be observed that the triangle ABC in the space diagram of Fig. 22 (a) and the force triangle of Fig. 22 (c) are similar triangles; therefore, their sides are proportional.

$$\frac{1{,}000}{5.77}=\frac{AB}{11.55}=\frac{AC}{10}$$

$$AB=2{,}000 \text{ lb, T. and } AC=1{,}730 \text{ lb, C.}$$

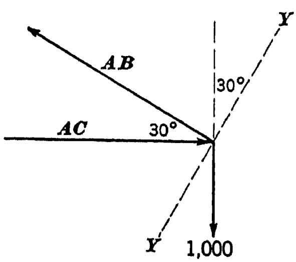


FIG. 23

Algebraic Solution.—In Fig. 23 sum forces perpendicular to AC or in a vertical direction.

$$\Sigma F_v=0$$
$$AB \sin 30^\circ-1{,}000=0$$
$$AB=2{,}000 \text{ lb, T.}$$

Sum forces perpendicular to AB or along YY. The force AB has no component along this line.

$$\Sigma F_y=0$$
$$AC \cos 60^\circ-1{,}000 \cos 30^\circ=0$$
$$AC=1{,}730 \text{ lb, C.}$$

PROBLEMS

19. A 1,000-lb weight is supported by two ropes making angles of 45° and 75° with the horizontal. Determine the stress in each rope. *Ans. 299 lb; 815 lb.*

20. Determine the stresses in members AB and AC of the crane shown in Fig. 24.

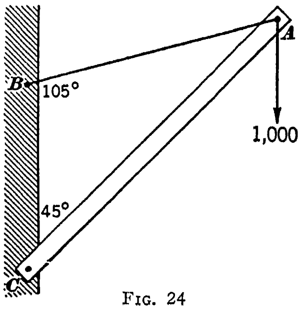


FIG. 24

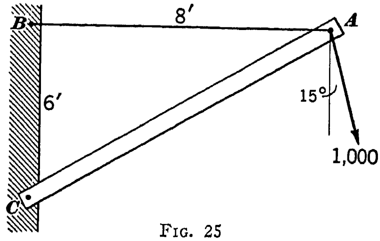


FIG. 25

22. Determine the force P and the stresses in the three ropes of Fig. 26.

23. Fig. 27 represents a 1,500-lb cylinder supported by two smooth planes at the points A and B. Determine the pressures on the cylinder at A and B.

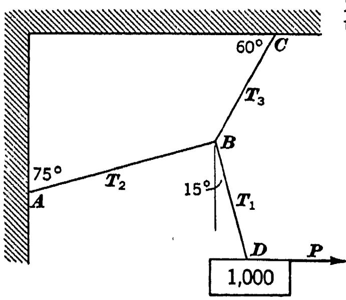


FIG. 26

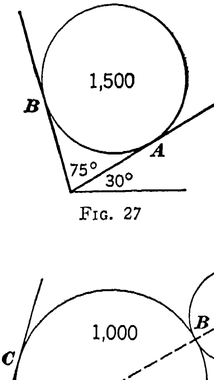


FIG. 27

24. A weight of 100 lb rests on a smooth plane and is prevented from moving by a 50-lb force acting upward at 60° with the horizontal. Determine the angle which the plane makes with the horizontal. *Ans.* 23.75°.

25. A 500-lb cylinder and a 1,000-lb cylinder rest in the box shown in Fig. 28. Determine the pressures at points A, B, C, and D.

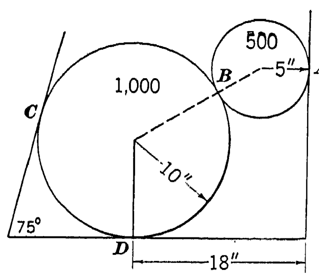


FIG. 28

19. Solution by Moments.—If a system of concurrent coplanar forces is in equilibrium, the resultant force is zero. The moment of the resultant force is zero; and by the principle of moments, Art. 16, the algebraic sum of the moments of all the component forces about any axis perpendicular to the plane of the forces is zero.

If the axis is taken through a point on the line of action of the unknown force, the moment of this force will be zero. This

unknown will thus be eliminated from the moment equation, and the equation will contain only one unknown. Another moment equation can be written by passing the axis through a point on the line of action of the other unknown. Two independent equations are thus obtained.

EXAMPLE 1

Solve for the stresses AB and AC of Fig. 29 (a) by the method of moments.

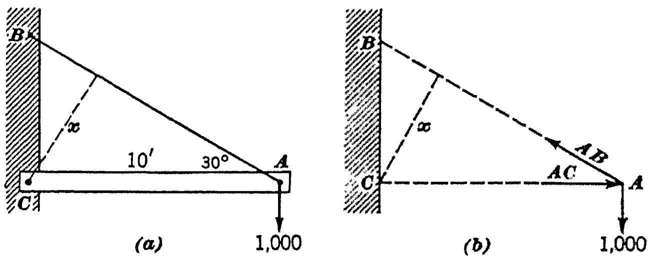


FIG. 29

$$x = 10 \sin 30° = 5$$
$$\Sigma M_C = 0$$
$$AB \times 5 - 1{,}000 \times 10 = 0$$
$$AB = 2{,}000 \text{ lb, T.}$$

$$\Sigma M_B = 0$$
$$AC \times 5.77 - 1{,}000 \times 10 = 0$$
$$AC = 1{,}730 \text{ lb, C.}$$

EXAMPLE 2

Solve for the stresses AB and AC in Fig. 30 (a) by several methods.

Graphical Solution.—Fig. 30 (b) is the free-body diagram and Fig. 30 (c) is the force triangle for the graphical solution.

Trigonometric Solution.—Since the angles are not given, they must be found from the dimensions of the structure. Applying the sine law to Fig. 30 (c) gives

$$\frac{1{,}000}{0.554} = \frac{AC}{0.866} = \frac{AB}{0.999}$$

$$AB = 1{,}800 \text{ lb, T. and } AC = 1{,}560 \text{ lb, C.}$$

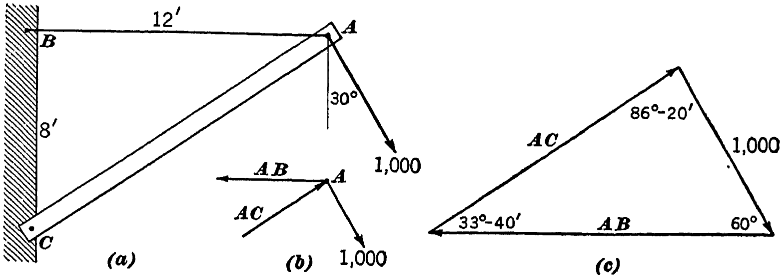


FIG. 30

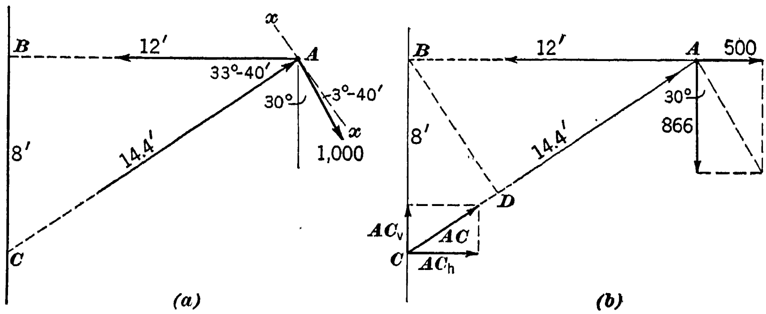


FIG. 31

Algebraic Solution.—In the free body for pin A, Fig. 31 (a), sum forces vertically and along the line xx, which is perpendicular to AC.

$$\Sigma F_v = 0 \qquad\qquad \Sigma F_x = 0$$

$$AC \times \frac{8}{14.4} - 1{,}000 \times 0.866 = 0 \qquad AB \times 0.555 - 1{,}000 \times 0.999 = 0$$

$$AC = 1{,}560 \text{ lb, C.} \qquad\qquad AB = 1{,}800 \text{ lb, T.}$$

Solution by Moments.—In some cases it is more convenient to work with the components of a force than with the resultant force. In the free body for pin A, Fig. 31 (b), the 1,000-lb force is resolved into horizontal and vertical components acting, as shown, at point A. The force AC representing the stress in member AC is moved to point C (principle of transmissibility of forces). At C this force is resolved into its horizontal and vertical components.

If the axis of moments is taken through point B, the horizontal component of the 1,000-lb force and the vertical component of the force representing the stress in member AC will pass through

B and therefore have no moment with respect to an axis through B. The same equation would result from using force AC and the moment arm BD.

$$\Sigma M_B = 0 \qquad\qquad \Sigma M_C = 0$$

$$AC \times \frac{12 \times 8}{14.4} - 866 \times 12 = 0 \qquad AB \times 8 - 866 \times 12 - 500 \times 8 = 0$$

$$AC = 1{,}560 \text{ lb, C.} \qquad\qquad AB = 1{,}800 \text{ lb, T.}$$

PROBLEMS

26. Use the method of moments for computing the stresses in the members of the wall crane shown in Fig. 24, in which member AB is 10 ft long. *Ans.* $AB = 1{,}414$ *lb, T.;* $AC = 1{,}932$ *lb, C.*

27. Using the method of moments, compute the stresses in the members of the crane illustrated in Fig. 25.

28. In Fig. 32 find the stresses in AB and AC by the moment method and check the results by a graphical solution.

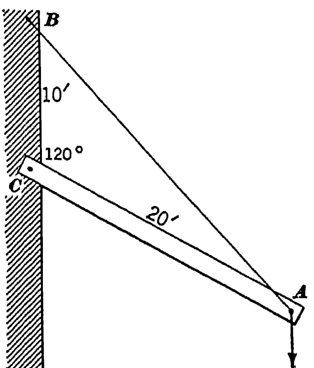


Fig. 32

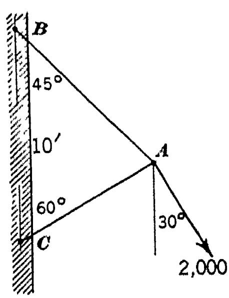


Fig. 33

29. Solve for the stresses in Fig. 33 by the moment method and check the results by two summations, each containing one unknown.

20. Body Held in Equilibrium by Three Coplanar, Non-Parallel Forces.

—Many problems arise in which a body is held in equilibrium by three coplanar, non-parallel forces. It can easily be shown that, if equilibrium is to be maintained, the three forces must intersect at a common point.

If the resultant of any two of the three forces is found, the third force must be equal, opposite, and collinear with this resultant, because the body is in equilibrium and the resultant of the three forces must be zero. Thus, it follows that the third force must

pass through the point of intersection of the other two forces, or that all three forces intersect at a common point.

This principle makes possible the solution of equilibrium problems when the lines of action of two forces and the point of application of a third force are known.

PROBLEMS

30. Determine the tensions in cords AB and CD, and also the direction of the cord CD, in Fig. 34. *Ans.* $AB = 56.5$ *lb;* $CD = 72.2$ *lb; 56.3°.*

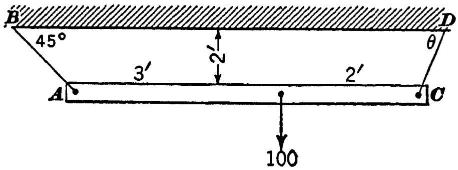


FIG. 34

31. Determine the force P, which will just start the wheel over the obstruction shown in Fig. 35.

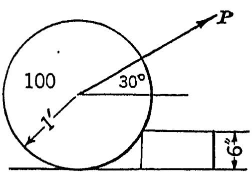


FIG. 35

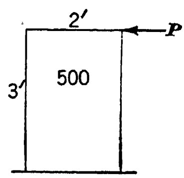


FIG. 36

32. In Fig. 36 the block rests on a plane which is sufficiently rough to prevent sliding. Determine the force P which will cause the block to tip.

REVIEW PROBLEMS[2]

33. Determine the amount and direction of the resultant of the following forces: 20 lb at 15° with the positive end of the X axis, 30 lb at 75°, 50 lb at 105°, and 200 lb at 240°. *Ans. 124.8 lb; 226.67°.*

34. Determine the amounts of forces P and Q in Fig. 37 by a graphical solution, and check the result by the summation method.

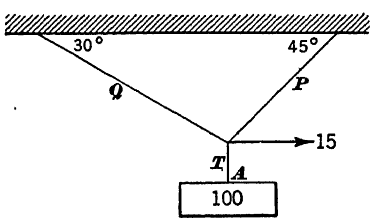


FIG. 37

[2] It is now suggested that the student review the problems of Chapter 2 and resolve a number of them by one or more of the alternative methods given in the illustrative examples. This procedure may seem a bit laborious, but those who follow it will be well paid for their efforts. Complete mastery of the basic principles presented up to this point is the only path to success in Mechanics.

35. Solve Problem 34 if the 15-lb force is removed and a 30-lb horizontal force acting to the right is applied at A tangent to the top of the 100-lb weight.

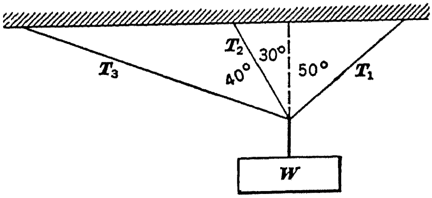


FIG. 38

36. If, in Fig. 38, T_1 is 150 lb and T_2 is 120 lb, compute W and T_3 which will be necessary for equilibrium of the weight.

37. Fig. 39 represents the pin at the left end of a roof truss. Solve for the stresses in members AB and AC. *Ans.* $AB = 10{,}000$ *lb, C.;* $AC = 8{,}660$ *lb, T.*

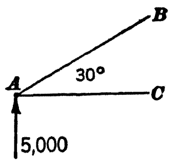


FIG. 39

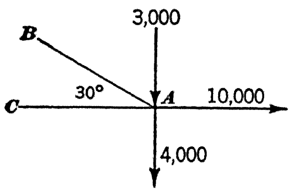


FIG. 40

38. Compute the values of forces AB and AC necessary to produce equilibrium in Fig. 40.

39. In Fig. 41, determine the values of forces AB and AC for equilibrium.

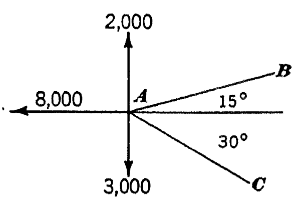


FIG. 41

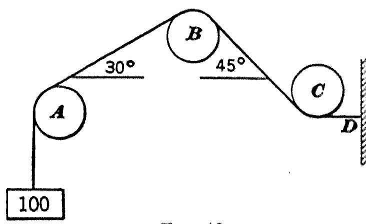


FIG. 42

40. Fig. 42 represents a 100-lb weight supported by a rope passing over three pulleys and fixed at D. Determine the amount and direction of the resultant bearing pressure at each of the pulleys A, B, and C.

41. Determine the pressures at A and B in Fig. 43 due to the weights of the two balls. *Ans.* $A = 3{,}000$ *lb;* $B = 1{,}294$ *lb.*

42. Solve for the pressures at A, B, and C due to the three balls shown in Fig. 44.

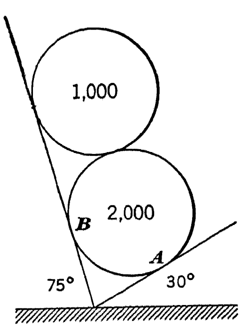


FIG. 43

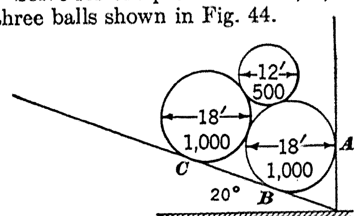


FIG. 44

43. Determine the pressures developed at points A, B, C, and D in Fig. 45 by the three spheres.

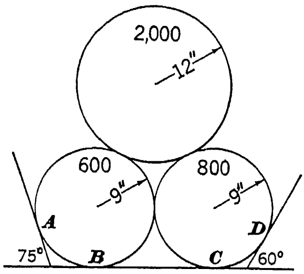


FIG. 45

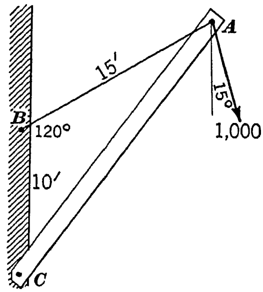


FIG. 46

44. In the crane of Fig. 46, determine the stresses in members AB and AC. Make the computation by each of the methods illustrated in this chapter.

45. Compute the stresses in the compression members of Fig. 47.

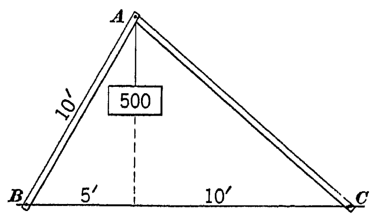


FIG. 47

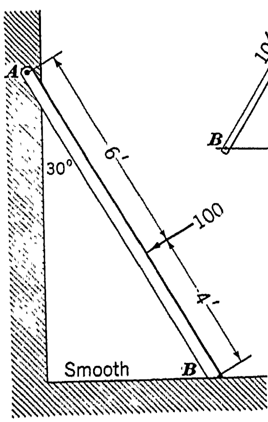


FIG. 48

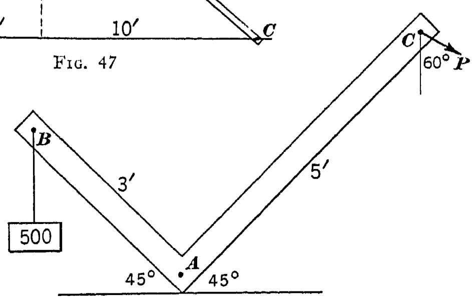


FIG. 49

46. What are the reactions at A and B caused by the 100-lb force in Fig. 48?

47. What force, acting upward at 45° above the horizontal, will prevent a 100-lb block from slipping down a 30° plane if the frictional resistance of the plane is 10 lb? *Ans. 41.4 lb.*

48. What are the amount and direction of the least force which can be applied to the block of Problem 47 to prevent slipping? Solve graphically.

49. Compute the force P and the amount and direction of the pin reaction at A, for equilibrium of the bell-crank lever shown in Fig. 49.

50. Compute the least force P for equilibrium in Problem 49.

51. The rectangular plate shown in Fig. 50 weighs 200 lb. It is supported as indicated at the corner A. Compute the tension in AB and the amount and direction of the force which must be applied at C, to maintain the plate in equilibrium as shown. *Ans.* $AB = 170$ *lb;* $CD = 144$ *lb.*

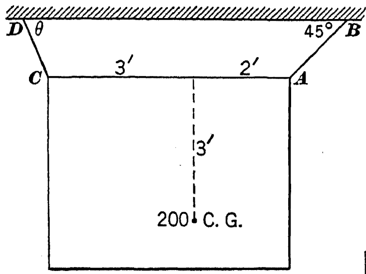


FIG. 50

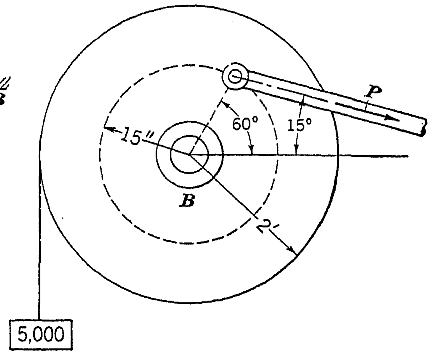


FIG. 51

52. Find the force P and the amount and direction of the bearing reaction at B for equilibrium of the hoisting drum in Fig. 51.

53. Two smooth planes in Fig. 52 support a weightless rod which carries a 1,000-lb load. Determine the angle θ for equilibrium when the rod is inclined at 15° with the horizontal.

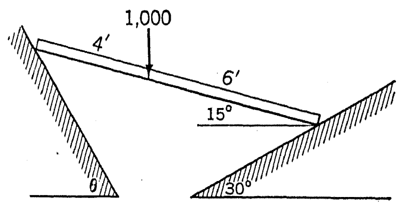


FIG. 52

54. Combine the loads and determine the magnitudes of the reactions at A and B, Fig. 53.

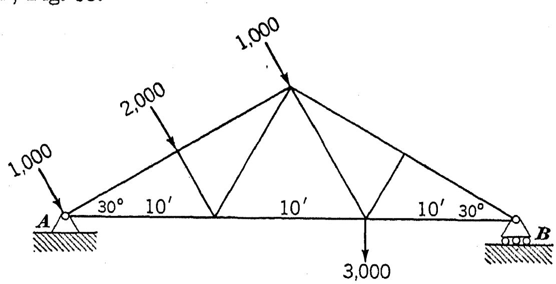


FIG. 53

55. Find the amount and direction of the resultant pin reaction at A caused by the 500-lb sphere in Fig. 54.

56. If both planes in Fig. 55 are smooth, determine the angle θ for equilibrium of the system.

57. Solve graphically for the tension in cord AB, Fig. 56, and the amount and direction of the reaction at pin C. Neglect the cross-sectional dimensions of beam BC.

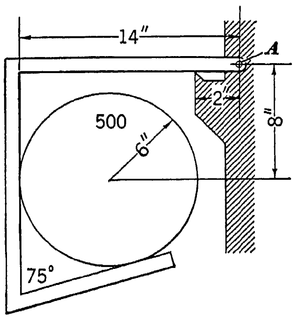


FIG. 54

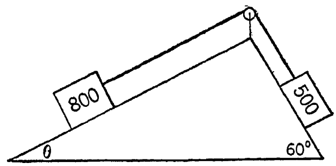


FIG. 55

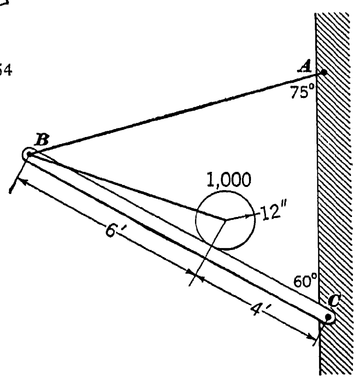


FIG. 56

CHAPTER 3

COPLANAR, PARALLEL FORCE SYSTEMS

21. **Bow's Notation.**[1]—Bow's Notation is a convenient means of designating forces and members of trusses or similar structures. The usual method is to start at the left end of the truss and, going around the outside in a clockwise sense, to place a lower-case letter in each space between the external forces; then to place one in each of the inside spaces in turn. A force or member is known by the letters in the spaces on each side. In Fig. 57 (*a*) the force over the left reaction is known as *ab*. On the load line, Fig. 57 (*b*), the lengths *AB*, *BC*, *CD*, *DE*, and *EA* represent, to a convenient scale, the magnitudes of the corresponding forces.

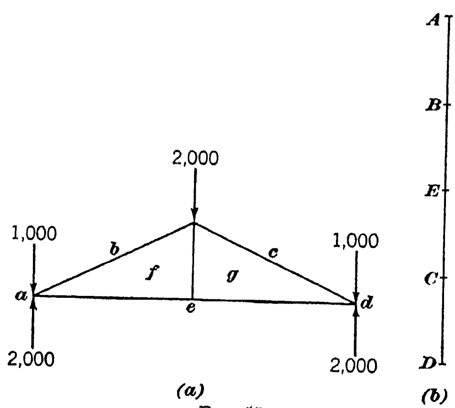


FIG. 57

22. **Resultant of Two Parallel Forces.**—The value of the resultant of two or more parallel forces is simply the vector sum of the forces or it is a couple (see Art. 28). The difficult part of the problem is to locate the line of action of the resultant force.

[1] For the remainder of this text, where graphical methods are used, the principles involved and the methods of solution will be developed before considering the analytical methods. The analytical methods will then be developed independently of the graphical work, so that the graphical work may be omitted if desired without affecting the coherence of the analytical work in any manner.

Since the forces do not meet, it is impossible to use the parallelogram or triangle law solution without modification.

In Fig. 58, by the parallelogram method, resolve forces F_1 and F_2 into components P_1, P_2, P_1', and P_2' so that P_1 and P_1' are equal and opposite and act along the same straight line; they will then cancel each other. Next produce the lines of action of components P_2 and P_2' until they intersect at point O. The components P_2 and P_2' are then moved to O (see Art. 4), and their resultant R is also the resultant of the original forces F_1 and F_2. This technique of canceling equal, opposite collinear components is the fundamental idea involved in most graphical solutions.

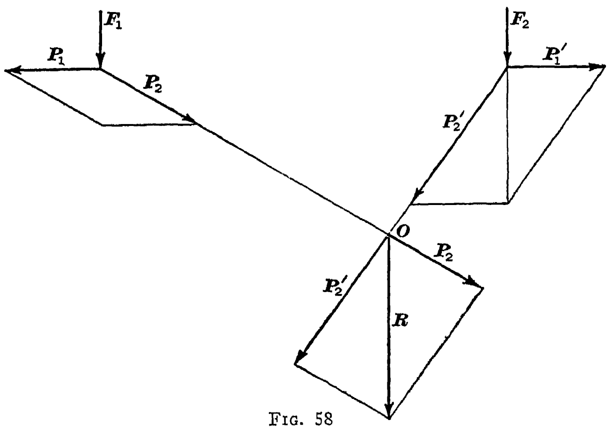


FIG. 58

PROBLEMS

58. Determine the resultant of a 100-lb force and a 70-lb force that is parallel to and distant 12 in. from the 100-lb force. *Ans. 170 lb; 4.93 in.*

59. In Problem 58, reverse the direction of the 70-lb force and determine the resultant.

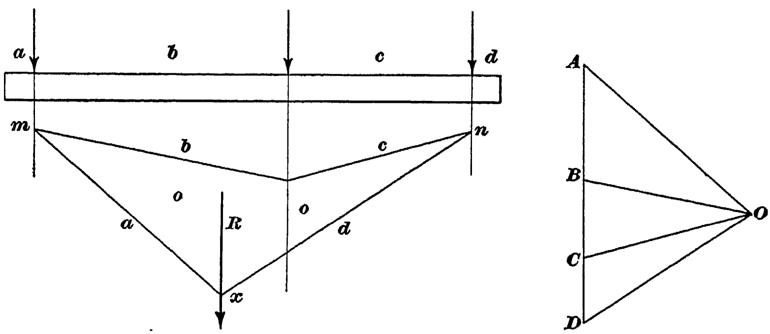
23. Resultant and Equilibrium of Parallel Force Systems of Three or More Forces.—The method of Art. 22 may be extended to determine the resultant of any number of forces but becomes rather involved. The method which follows depends on the same principles but requires a much less laborious construction.

EXAMPLE 1

Locate the line of action of the resultant of the three forces shown acting on the beam of Fig. 59.

Using Bow's Notation, lay off to scale on the load line, Fig. 59 (*b*), the forces AB, BC, and CD. From any convenient point O draw the rays AO, BO, CO, and DO.

In Fig. 59 (*a*), starting from any convenient point m on the line of action of force ab, draw a line ob parallel to ray OB of Fig. 59 (*b*) and intersecting the line of action of force bc; from this point, draw co parallel to CO and intersecting cd at point n. Through the points m and n, draw ao and do parallel, respectively, to AO and DO and intersecting at point x. The resultant of the force system passes through x and is parallel to the given forces.



Funicular Polygon (*a*) Force Diagram (*b*)

FIG. 59

If the construction of Fig. 59 is studied, it will be found to be based on the principle of cancellation of components illustrated in Art. 22. In the force diagram in (*b*), force AB is resolved into the components AO and OB. The component OB acts to the left. Force BC is resolved into the component BO (equal and opposite to OB) and the component OC acting to the left. Force CD is resolved into CO (equal and opposite to OC) and the component OD acting to the left. The equal and opposite components cancel each other, and AO and OD are the only remaining components of $AB \leftrightarrow BC \leftrightarrow CD$.

The strings ao and ob of the funicular polygon, Fig. 59 (*a*), determine the lines of action of the components AO and OB of the force AB, which acts along the line ab. Similarly, the lines of action of the components of the other forces are determined by the other strings of the funicular polygon. Since all components except AO (acting along ao) and OD (acting along od)

are canceled, the intersection of *ao* and *od* at *x* in Fig. 59 (*a*) determines a point on the line of action of the resultant force *R*, which is equal and parallel to *AD* in Fig. 59 (*b*).

Should point *D* of Fig. 59 (*b*) coincide with point *A*, the resultant of the system would be either a force of zero magnitude or a couple (see Art. 28). If the system reduces to a couple, the two strings *ao* and *od* of the funicular polygon will be parallel lines. The magnitude or moment of the couple will then be determined by the product of one of the equal, opposite, and coincident forces *AO* and *OD* of the force polygon in Fig. 59 (*b*) and the perpendicular distance between the parallel strings *ao* and *od* of the funicular polygon in Fig. 59 (*a*). Should the strings *ao* and *od* be coincident, the moment arm of the couple will be zero and the funicular polygon will close. Therefore, when the force polygon and the funicular polygon are closed figures, $R=0$ and $M=0$, and the system is in equilibrium.

The most common problem involving parallel force systems is not the determination of the resultant, but is rather the magnitude of certain forces necessary to produce equilibrium.

If a system of parallel forces is in equilibrium, the resultant must be zero; that is, $R=0$. There must also be no tendency to rotate; hence, $\Sigma M=0$. Graphically these conditions are satisfied when:

(*a*) Force polygon is a closed figure, or $R=0$.
(*b*) Funicular polygon closes, or $\Sigma M=0$.

EXAMPLE 2

The truss shown in Fig. 60 (*a*) is held in equilibrium by the three known loads and the two unknown reactions R_1 and R_2. Determine the amounts of the reactions.

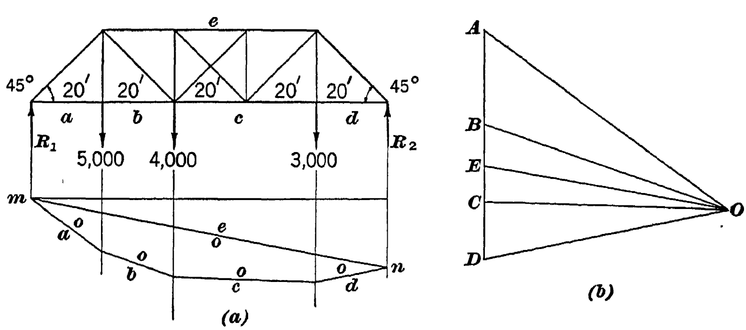


FIG. 60

The known forces AB, BC, and CD are laid off to scale on the load line of Fig. 60 (*b*). If the truss is in equilibrium, the unknown reactions DE and EA must close the force polygon. The location of the point E is unknown. Its location can be determined because, for equilibrium of the truss, $\Sigma M=0$ and the funicular polygon must close. Select any convenient point O and draw the rays OA, OB, OC, and OD as shown in Fig. 60 (*b*).

The funicular polygon in Fig. 60 (*a*) can now be started at any convenient point, such as m on the line of action of R_1. If, in any case, the line of action of one reaction is undetermined, then the funicular polygon must be started at the point of application of this unknown force, since that is the only known point on its line of action. From m the string oa of the funicular polygon is drawn parallel to the ray OA of the force diagram. Starting at the point where oa intersects the line of action of force ab, string ob is drawn parallel to OB of the force diagram. The strings oc and od are drawn in a similar manner. The string od intersects the line of action of R_2 at n.

Since for equilibrium $\Sigma M=0$ and the funicular polygon must close, oe must connect points m and n. Then the two remaining uncanceled components, OE of R_1 and EO of R_2, will be equal, opposite, and collinear. If a line is drawn through the point O in the force polygon, Fig. 60 (*b*), parallel to string oe, the point E on the load line will be determined. The reaction R_1 is represented to scale by the length of vector EA, and R_2 by the length of vector DE. In this example, $R_1=7{,}000$ lb and $R_2=5{,}000$ lb.

PROBLEMS

60. In Fig. 59 (*a*) force ab is 100 lb, bc is 200 lb, and cd is 300 lb. The distances between the forces are, respectively, 10 and 15 in. Compute the amount and the position of the resultant. *Ans. 600 lb; 15.83 in.*

61. In Fig. 60 (*a*) replace the 4,000-lb load by a load of 10,000 lb located one span to the right. Get reactions.

24. Resultant of Two or More Parallel Forces by Inverse Proportion.—In Fig. 61, F_1 and F_2 are parallel forces of the same sense, and $R=F_1+F_2$ is the resultant. Draw any line mn. From m lay off to scale mp equal to F_2; and from n lay off to scale nq equal to F_1 reversed. Draw pq which intersects mn at O. The point O is on the line of action of R. The proof of this construction depends on the principle of moments. Draw through O lines

a and b perpendicular to F_1 and F_2. Fig. 61 consists of pairs of similar triangles, since the angles of each pair are equal. Hence,

$$\frac{F_1}{F_2}=\frac{b}{a}=\frac{b'}{a'}$$

According to the principle of moments, if point O is on the line of action of the resultant, the moment of F_1 about an axis through O equals the moment of F_2 about that axis since the moment of R must be 0. The foregoing equation satisfies these conditions, because $F_1a=F_2b$. Therefore, R must pass through point O.

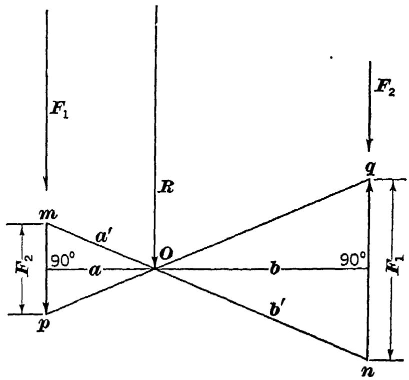


Fig. 61

The preceding equation also tells us that F_1 and F_2 are inversely proportional to their perpendicular distances from R; and also are inversely proportional to any diagonal distances from R, such as a' and b'.

This method may be extended to locate the resultant of any number of parallel forces by finding the resultant of any two forces; then combining this resultant with a third parallel force; and continuing until the force system is reduced to a single resultant.

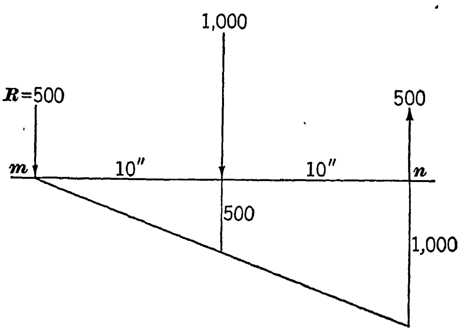


Fig. 62

If the two forces which are to be combined are opposite in direction, the construction is similar; but both forces are laid off in the same direction from the base line mn, as in Fig. 62, and not in opposite directions as in the preceding case.

PROBLEMS

62. In Fig. 63 determine the resultant of the 4,000- and 6,000-lb forces.

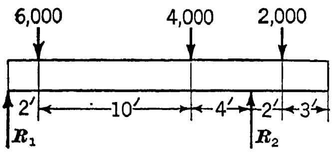


Fig. 63

63. Combine the 2,000-lb force of Fig. 63 with the resultant from Problem 62.

64. Reverse the direction of the 4,000-lb force in Fig. 63, and determine the resultant of the three forces.

25. Resolution of a Force Into Two Parallel Components. By reversing the procedure of Art. 24, a force may be resolved into two parallel component forces, acting along any two lines parallel to the original force. In Fig. 64, EF represents to scale

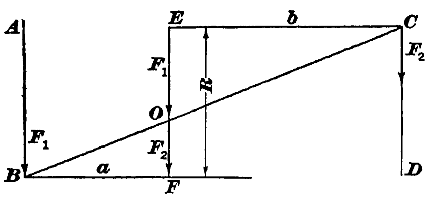


Fig. 64

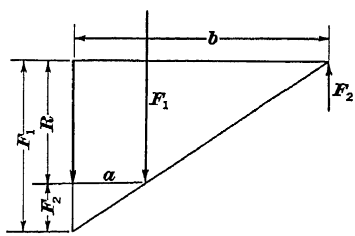


Fig. 65

the force to be resolved into components along the lines AB and CD. Through the ends of vector EF draw the lines CE and BF perpendicular to EF. Connect C and B. The point O on CB divides EF into two components, EO acting along AB, and OF acting along CD. From the similar triangles EOC and BOF,

$$\frac{F_1}{F_2}=\frac{b}{a} \text{ and } F_1a=F_2b$$

If the resultant force does not lie between the two component forces, the construction shown in Fig. 65 is used. Here,

$$\frac{F_1}{F_2}=\frac{b}{a} \text{ and } F_1a=F_2b$$

As explained in Art. 24, it is not necessary that the lines a and b be perpendicular to the forces; but a and b must be parallel lines.

PROBLEMS

65. Resolve the resultant of Problem 63 into components along the lines R_1 and R_2 of Fig. 63. *Ans. 6,000 lb; 6,000 lb.*

66. Resolve each of the forces of Fig. 63 into components along R_1 and R_2. Add these components and compare with Problem 65.

67. Resolve a downward 2,000-lb force into components, F_1 and F_2, along lines 10 in. and 15 in. to the right of the 2,000-lb force.

26. Resultant of Any Number of Coplanar Parallel Forces Algebraically.—The numerical value of the resultant of a system of coplanar parallel forces is the algebraic sum of the component forces. By the principle of moments, the moment of this resultant with respect to any axis perpendicular to the plane of the forces must be equal to the algebraic sum of the moments of the component forces with respect to the same axis.

Writing moments about an axis through any point O, Fig. 66, the following equation is obtained:

$$Rr = F_1d_1 - F_2d_2 + F_3d_3$$

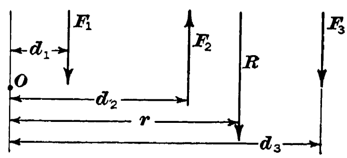


FIG. 66

PROBLEMS

68. Locate the resultant of the three loads shown in Fig. 63. *Ans. 12,000 lb, 8 ft from R_1.*

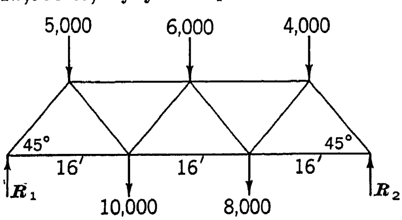


FIG. 67

69. If the 4,000-lb force of Fig. 63 is reversed, where does the resultant act?

70. Determine the resultant of the three loads on the top of the truss shown in Fig. 67.

71. Determine the resultant of all the loads on the truss of Fig. 67.

27. Equilibrium of Coplanar, Parallel Force Systems.—If a coplanar, parallel force system is in equilibrium, the resultant of the system must be zero, since there can be no tendency to accelerate in a direction parallel to the lines of action of the forces. There also must be no tendency to rotate. Thus, there are two conditions which must be satisfied to produce equilibrium of a coplanar, parallel force system, or $\Sigma F = 0$ and $\Sigma M = 0$.

Since only two independent equations can be written, there cannot be more than two unknown forces if a solution is to be obtained.

EXAMPLE

Determine the reactions R_1 and R_2 necessary to support the beam shown in Fig. 68.

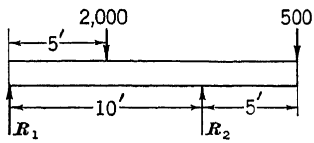


Fig. 68

$$\Sigma M_{R_2}=0$$
$$-10R_1+2{,}000\times 5-500\times 5=0$$
$$R_1=750 \text{ lb}$$
$$\Sigma F_v=0$$
$$R_1+R_2-2{,}500=0$$
$$R_2=2{,}500-750=1{,}750 \text{ lb}$$

The reaction R_2 may also be determined by writing a second moment equation with the axis of moments through a point on R_1. Then the equation $R_1+R_2-2{,}000-500=0$ offers a means of checking the accuracy of the results.

PROBLEMS

72. Determine the amount and location of the single force necessary to produce equilibrium in Fig. 68. *Ans. 2,500 lb, 7 ft from R_1.*

73. In Fig. 68, if R_1 is unknown and R_2 acts at the right end, determine the values of R_1 and R_2 for equilibrium.

74. In Fig. 69, the beam weighs 100 lb per ft; and, in addition to the 2,000-lb concentrated load, it has a uniformly distributed load of 200 lb per ft, extending 10 ft from the right end. Considering the distributed loads as acting at their centers of gravity, determine R_1 and R_2 for equilibrium.

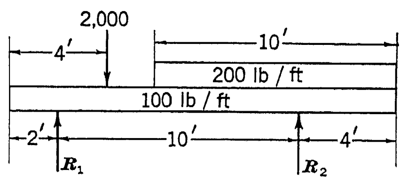


Fig. 69

75. In Fig. 69 let R_2 act at the right end of the beam, and add a 5,000-lb concentrated load at a point 6 ft from the right end. Determine the reactions necessary for equilibrium.

28. Couples.—A couple consists of two equal, oppositely directed and parallel forces. Since the vector sum of such forces is zero, they can produce no direct or resultant force effect. The only effect which can be produced by a couple is to cause a positive or negative torque or turning moment to be applied to the rigid body on which the couple acts.

(1) The turning moment of any given couple is a constant and is always equal to the product of one of the parallel forces and the perpendicular distance between the forces. The turning moment is independent of the location of the axis of moments.

Let O, Fig. 70, be any point in the plane of the couple consisting of the two forces P, and let d_1 and d_2 be the perpendicular distances from O to the lines of action of the forces P. The moment equation for an axis through O is

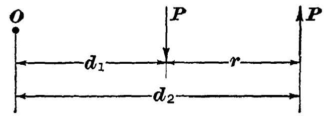


FIG. 70

$$Pd_2 - Pd_1 = M \text{ or } P(d_2 - d_1) = Pr = M$$

This indicates that the turning moment about the axis through O or any parallel axis is a constant Pr.

(2) A couple may be transferred to any plane parallel to its original plane without changing its effect. This is evident from the discussion under (1).

(3) A couple may be replaced by any other couple which has the same moment and sense. The magnitude of the forces or the distance between the forces and their positions in the plane of the couple or in any parallel plane may be varied at will, provided the magnitude of the couple remains unchanged. This also follows from the discussion in (1).

(4) A single force cannot balance or cause equilibrium of a couple. Since a couple consists of two equal and opposite forces, the addition of any single force cannot make the sum of the forces equal to zero. The only manner in which this sum can remain zero is to add two equal and opposite forces, or another couple. If the added couple has a moment equal and opposite to the original couple, the system will be placed in equilibrium.

(5) The resultant moment of any number of coplanar couples or couples in parallel planes is simply the algebraic sum of their moments. This is axiomatic.

(6) Couples can be represented by vectors. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The length of the vector represents the magnitude of the moment to scale. The arrow of the vector should point in the direction toward which a right-hand screw would travel if turned by the given couple. See Fig. 71.

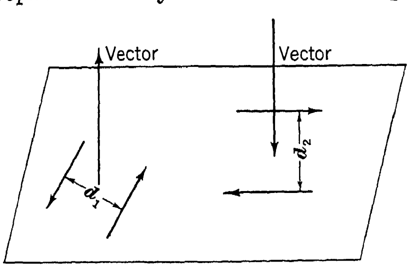


FIG. 71

PROBLEMS

76. Determine the turning moment of the 30-lb forces about an axis through the center of the bar at A in Fig. 72. If the 20-in. dimension is changed to 16 in., what is the turning moment about an axis through A?

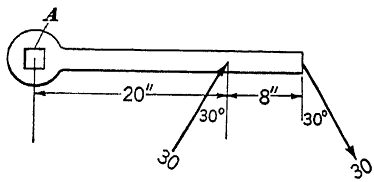


FIG. 72

77. If a similar wrench is attached to the other end of the bar in Fig. 72, but with the resisting forces acting normal to the wrench and at 15 and 25 in. from the center of the bar, what forces will be required for equilibrium of the bar?

78. The vertical plate in Fig. 73 is attached to a horizontal shaft A. Determine the resultant torque which must be applied to the shaft A for equilibrium. Represent this torque by means of a vector to a scale of 100 lb to the inch.

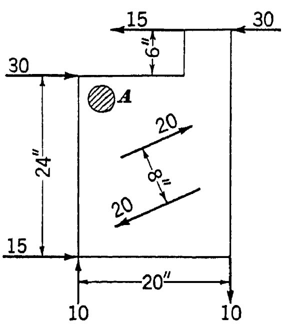


FIG. 73

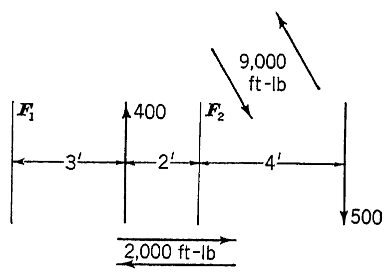


FIG. 74

79. Determine the magnitude and sense of the forces F_1 and F_2 for equilibrium of the system in Fig. 74.

29. Resolution of a Force Into a Force at a Chosen Point and a Couple; and, Conversely, Combination of a Force and a Couple Into a Single Force.—It is often convenient and clarifying to resolve a force into a force parallel to the given force and a couple in the plane of the force. In Fig. 75 (a), P is the given force acting at the edge of the post and O, the midpoint of the post, is the chosen point. The two equal, opposite, and collinear forces $P_1 = P_2 = P$ are placed at O. The total load on the post remains

unchanged. Now P and P_1 form a clockwise couple whose moment about an axis through O is the same as the moment of the original force P, and there is also the downward load $P_2 = P$ on the post. The only effect on the post has been to change the

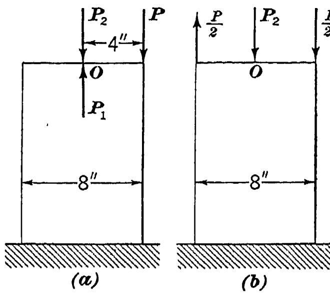


FIG. 75

line of action of the downward load from the edge of the post to the center O. Since it is possible to move a couple around in its plane, the couple may be transferred as in Fig. 75 (b), or in any convenient manner which does not change its magnitude or sense, without any change in the loading of the post.

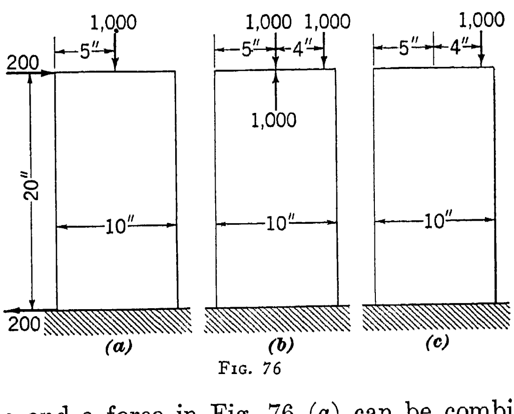


FIG. 76

A couple and a force in Fig. 76 (a) can be combined into a single force in the following manner: The value of the given couple is $200 \times 20 = 4{,}000$ in.-lb. A couple of $1{,}000 \times 4 = 4{,}000$ in.-lb is equivalent to the original couple. In Fig. 76 (b) the equivalent couple is shown placed so that one of its forces is collinear with the original downward 1,000-lb load. These equal and collinear forces cancel, leaving only the 1,000-lb downward force acting 4 in. from the center of the post, as in Fig. 76 (c).

PROBLEMS

80. Resolve the 10-lb force acting on the steering wheel, Fig. 77, into a single force acting at the center and a couple consisting of horizontal tangential forces.

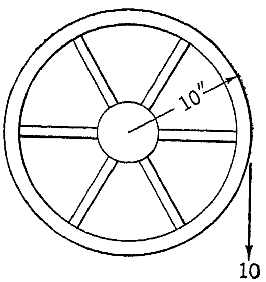


FIG 77

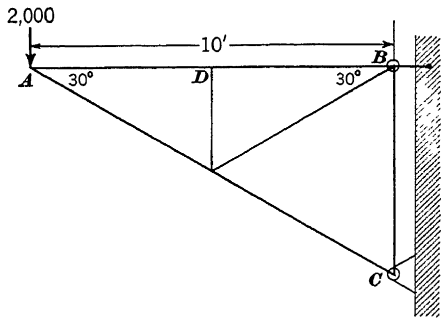


FIG. 78

81. Replace the 2,000-lb load in Fig. 78 by a force and a couple which produce the same effect on the pins at B and C. The couple forces are to act at A and D.

82. Replace the 500-lb load and the couple in Fig. 79 by a single 500-lb vertical force. The bar is supported by a bearing at A.

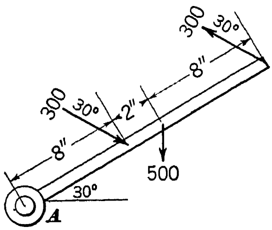


FIG. 79

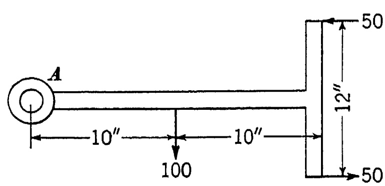


FIG. 80

83. The bar in Fig. 80 is supported at bearing A. Locate the single 50-lb force which will produce the same effect as the loads shown.

REVIEW PROBLEMS

84. Downward forces of 100 and 200 lb are acting 12 in. apart. Determine the resultant. *Ans. 300 lb, 8 in. from 100 lb.*

85. Downward forces of 150, 75, and 200 lb are 3 ft and 6 ft apart. Determine the resultant.

86. If the 75-lb force of Problem 85 is reversed, what is the resultant of the system?

87. A 4,000-lb automobile has a wheel-base of 120 in. If the rear wheels carry 2,500 lb and the front wheels 1,500 lb, where might the 4,000-lb weight be considered concentrated without change in the wheel reactions?

88. Determine the resultant of the loads on the beam shown in Fig. 81. *Ans. 11,000 lb, 10.55 ft from R_1.*

89. Determine the reactions R_1 and R_2 for equilibrium of the beam in Fig. 81.

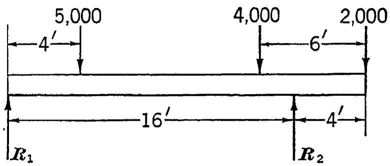


FIG. 81

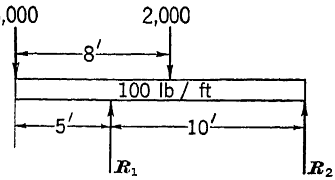


FIG. 82

90. What are the values of the reactions in Fig. 82 for equilibrium?

91. What force must be applied at a point 2 ft from the right end of the beam shown in Fig. 82, if the right reaction is to be 1,000 lb in an upward direction? *Ans. 1,906 lb.*

92. By the graphic method of inverse proportion, determine the resultant of the loads in Fig. 81.

93. By the graphic method of inverse proportion, determine the reactions of the beam of Fig. 81.

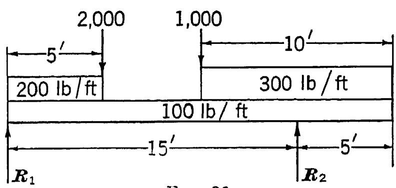


FIG. 83

94. Find the reactions for the beam shown in Fig. 83.

95. Find the reactions for the truss shown in Fig. 84. *Ans.* $R_1 = 10{,}666$ *lb;* $R_2 = 12{,}334$ *lb.*

96. By inspection, and the method of inverse proportion, solve for the resultant of the loads in Fig. 84.

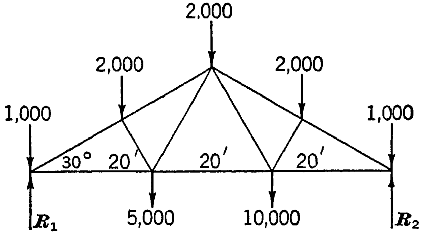


FIG. 84

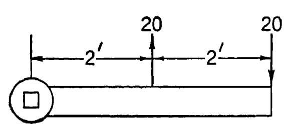


FIG. 85

97. What is the resultant turning moment exerted by the wrench shown in Fig. 85? Explain the result.

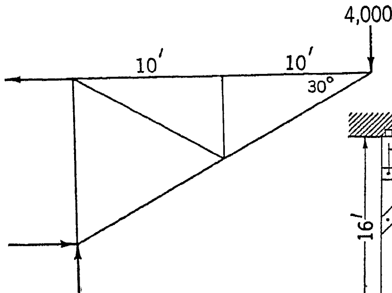


FIG. 86

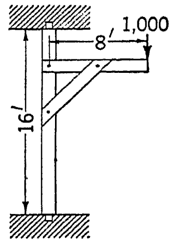


FIG. 87

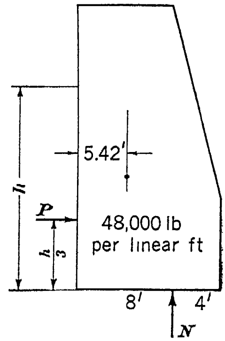


FIG. 88

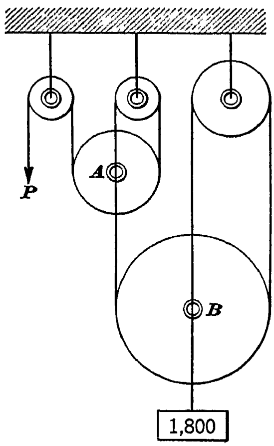


FIG. 89

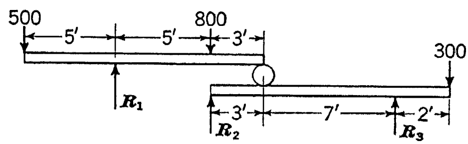


FIG. 90

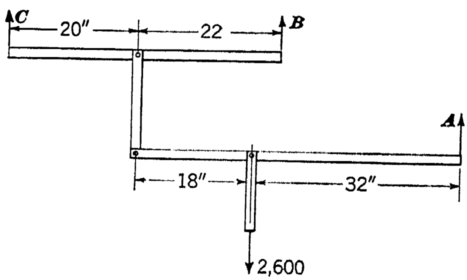


FIG. 91

98. Fig. 86 represents a cantilever truss supported by a vertical force and a couple. Determine the values of the force and the couple.

99. Fig. 87 represents a crane with plane bearings at the floor and ceiling so that the crane may be turned through 360°. Show the couples acting, and determine the values of the forces which form the couples. *Ans. 1,000 lb; 500 lb.*

100. Fig. 88 represents a cross-section of a concrete dam. If it is assumed that the upward reaction of the ground acts at N, and that P, the resultant water pressure, acts at one-third the depth of the water from the base, what couples are acting? Determine the maximum height to which the water can rise if equilibrium is to be maintained. Concrete weighs 150 lb per cu ft.

101. If all sheaves in Fig. 89 are frictionless, determine the force P required to support the 1,800-lb load. At A and B each rope is attached to the sheave axle.

102. Compute R_1, R_2, and R_3 for equilibrium of Fig. 90.

103. Fig. 91 represents a type of hitch which may be used for a three-horse team. If the drawbar pull of the vehicle is 2,600 lb, how much force does each of the horses supply at A, B, and C?

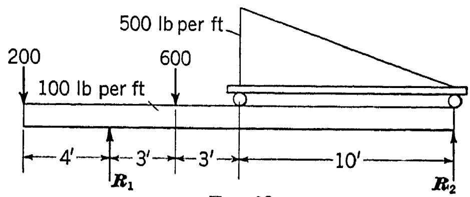


Fig. 92

104. Compute the reactions R_1 and R_2 in Fig. 92.

105. Determine the weight W, Fig. 93, and the horizontal and vertical pin reaction at bearing A for equilibrium, if all bearings are frictionless.

106. Solve for the reactions at pins A and G in Fig. 94.

107. Determine the angle θ for equilibrium of the bell-crank lever shown in Fig. 95.

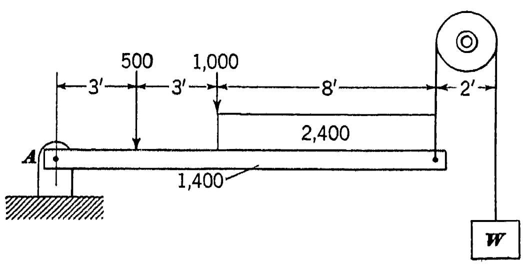


Fig. 93

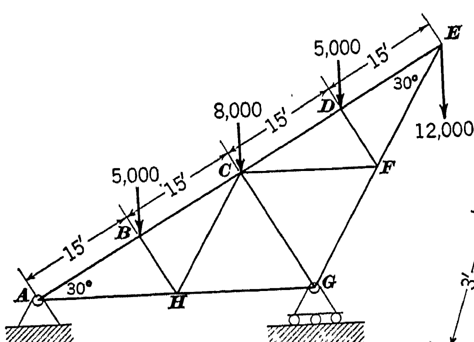


Fig. 94

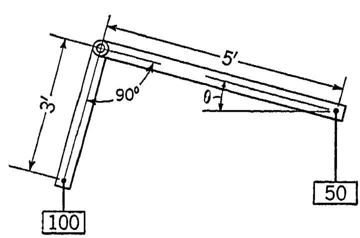


Fig. 95

CHAPTER 4

COPLANAR, NON-CONCURRENT FORCE SYSTEMS BY GRAPHICAL METHODS

30. Definition.—A coplanar, non-concurrent force system consists of several forces, all of which have their lines of action in a common plane but which do not meet in a common point.

31. Resultant of Coplanar, Non-Concurrent Force System by Parallelogram Method.—The resultant of a coplanar, non-concurrent force system is the single force or couple which will produce the same effect as the several forces acting together.

EXAMPLE

Determine the resultant of the forces shown in Fig. 96 by the parallelogram method.

Extend the lines of action of the 100- and 200-lb forces until they intersect at O. The parallelogram construction gives $R_1 = 159$ lb. Produce the lines of action of R_1 and the 300-lb force until they intersect at N. By a second parallelogram determine $R_2 = 447$ lb, which is the resultant of the system. This method may be extended to any number of forces.

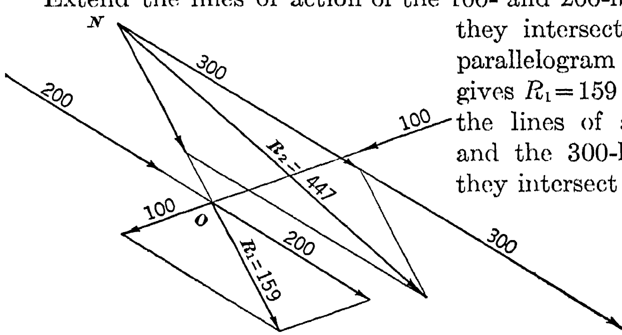


FIG. 96

PROBLEM

108. Fig. 97 shows a piece of timber acted upon by three forces. Determine the amount and direction of the resultant force by the method of Art. 31. *Ans. 494 lb; 10.95°.*

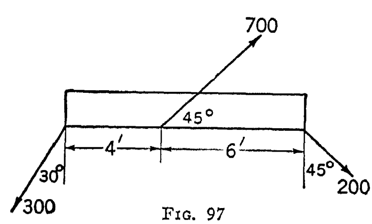


FIG. 97

32. **Resultant of a Coplanar, Non-Concurrent Force System by Funicular Polygon Method.**—The method developed in Art. 23 for parallel forces may be used for coplanar, non-parallel force systems.

EXAMPLE

Determine the resultant of the system of forces shown in Fig. 98 (*a*) by the funicular polygon method.

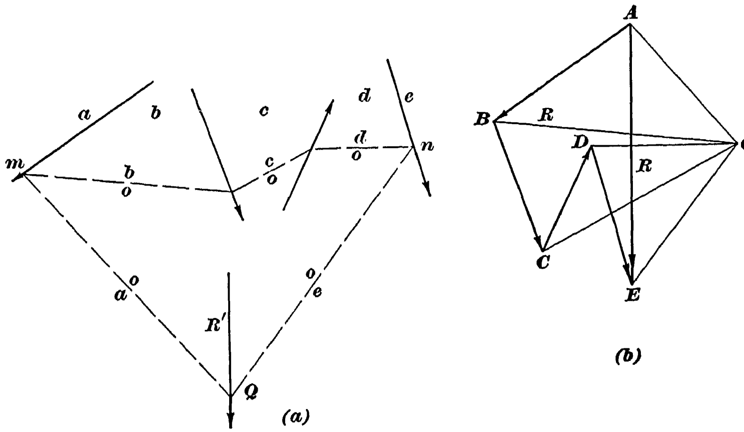


FIG. 98

Lay down the force polygon in Fig. 98 (*b*) to scale. The force R is the resultant of the given force system in amount and direction. Its position or line of action must be determined by means of the funicular polygon. Select any convenient pole O, and draw the rays AO, BO, CO, DO, and EO, Fig. 98 (*b*). Starting at any point m on force ab, Fig. 98 (*a*), draw ob parallel to OB, oc parallel to OC, and od parallel to OD, intersecting de at n. Through m draw oa parallel to OA, and through n draw oe parallel to OE. The lines oa and oe intersect at Q. The resultant R' is drawn through Q, parallel to R in Fig. 98 (*b*).

This construction is based on the cancellation of components as explained in Art. 23. The system is reduced to the components AO and OE, Fig. 98 (*b*). The vector sum of these components is $AO \leftrightarrow OE = R$. If lines are drawn through the points of resolution m and n, parallel to AO and OE, they intersect at Q, which is a point on the line of action of the resultant force R.

A force system may reduce to a couple and not to a single force. In this case the force polygon made from the given forces

will form a closed figure, as *the algebraic sum of the system is zero.* The funicular polygon will not close; and the *last two strings will be parallel lines,* indicating that the system has been reduced to two equal, opposite, and parallel forces, or a couple, the magnitude of which is determined as in Art. 23.

PROBLEMS

109. Determine the resultant of the wind and dead loads on the truss of Fig. 99 by the funicular polygon method. Check the result by the parallelogram method of Art. 31. *Ans. 31,340 lb, 13.8 ft from* R_1.

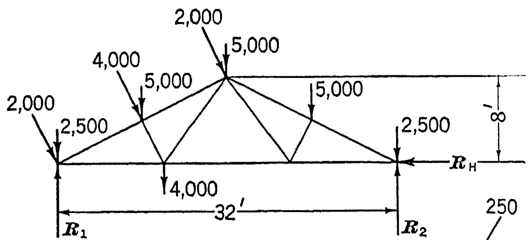


FIG. 99

110. By the funicular polygon method determine the resultant of the force system of Fig. 100. Discuss the result

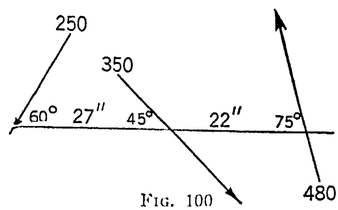


FIG. 100

33. Equilibrium of Coplanar, Non-Concurrent Force Systems.

Graphically, the conditions for equilibrium of a coplanar, non-concurrent force system are, by Art. 23:

(*a*) The force polygon must close, as $\Sigma F = 0$;

(*b*) The funicular polygon must close, as $\Sigma M = 0$.

If the force polygon is a closed figure, the resultant force $R = 0$, or the sum of the components of the forces along each of any two intersecting lines is zero. This gives two independent conditions of equilibrium.

If the funicular polygon is a closed figure, the resultant moment is zero, or $\Sigma M = 0$. This is a third condition of equilibrium.

Thus a coplanar, non-concurrent force system has three independent conditions of equilibrium and may have three unknown quantities; and yet a definite solution may be made. The unknown quantities may be the amounts of three forces, the directions of three forces, the amount and direction of one force together with the amount or direction of a second force, or any similar combination.

EXAMPLE

Determine the forces required at A and B, Fig. 101 (a), for equilibrium of the cantilever truss.

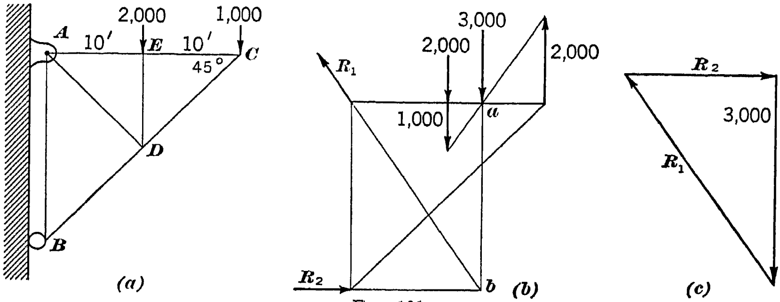


FIG. 101

The simplest solution of the problem is by the *force triangle method.* By inverse proportion, combine the 1,000- and 2,000-lb loads. This gives 3,000 lb acting at a in Fig. 101 (b). The truss is now held in equilibrium by the action of the three forces R_1, R_2, and the 3,000-lb resultant. These three forces must intersect at a common point, according to Art. 20. Since R_2 is horizontal, R_2 and the 3,000-lb force intersect at point b, and thus the direction of R_1 is determined. Fig. 101 (c) gives the force triangle for determining the values of R_1 and R_2.

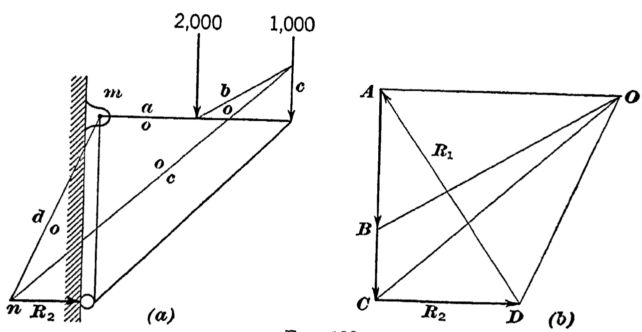


FIG. 102

Solution by the Funicular Polygon Method.—In Fig. 102 (b), lay off AB and BC to scale equal, respectively, to 2,000 and 1,000 lb. Through C draw a line parallel to R_2. The force polygon cannot be closed, as the direction of R_1 is not known. From any point O, draw rays OA, OB, and OC. Since m in Fig. 102 (a)

is the only point known on R_1, the string or funicular polygon must start at m (see Art. 23). Draw ao, bo, and co parallel to AO, BO, and CO. The string co intersects force cd or R_2 at n. For equilibrium the string polygon must close, so that a line connecting m and n is the closing line (see Art. 23). The string od is parallel to the unknown ray OD of the force diagram. Through O in the force diagram, draw OD parallel to od of the funicular polygon. This locates the point D of the force diagram and thus determines the amounts of the reactions R_1 and R_2.

PROBLEMS

111. If the truss of Fig. 99 has a roller under the left end and a hinge or a pin at the right reaction, determine the reactions by the funicular polygon method. Hint: Start the funicular polygon at the hinge. *Ans.* $R_1 = 17{,}675$ *lb;* $R_{2H} = 3{,}580$ *lb;* $R_{2V} = 13{,}485$ *lb.*

112. Check the results of Problem 111 by combining all the forces into a single resultant force, and then by inverse proportion resolve this resultant into the two reactions.

113. Solve for the reactions at A and B in the truss shown in Fig. 101 (a), if the member AB is removed and the truss is attached to the wall by pins at A and B, instead of the pin and roller shown in Fig. 101 (a).

34. **Trusses.**—A truss is a structure made up of straight bars joined together at the ends by pins in such a manner that the bars form triangles.

Actually trusses are usually joined together by rivets or welding rather than by pins or bolts at the joints. If the design and workmanship of construction are such that the lines of action of all the forces exerted by the several members meeting at a joint intersect at a common point, it is permissible to treat the joint as pin connected. The assumption of pin connected joints means no twisting at the joints. Each joint is a pure coplanar, concurrent force system. Equilibrium of such a system requires only that $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

A triangle is a rigid or stable figure and cannot be distorted without changing the lengths of its sides. All the external loads are applied to the truss at the pins which join the triangular units together. When the loads are applied in this manner, the members which form the triangular units are not subjected to any bending action, but carry only direct tensile stress or direct compressive stress. *The line of action of the stress in any member is, therefore, always along the line connecting the pins at the two ends of the member.*

35. Stresses by Method of Joints.—Each pin or joint of a truss is acted upon by a coplanar, concurrent force system, and the stresses may be computed by solving such force systems.

The conditions for equilibrium of such a system are $\Sigma F_x=0$ and $\Sigma F_y=0$. Therefore, it is not possible to use the method of joints at pins where there are more than two unknown stresses, since there are only two conditions of equilibrium to be satisfied.

EXAMPLE

Solve for the stress in each member of the truss shown in Fig. 103.

The left reaction is vertical, because of the roller. The direction of the right reaction must be determined.

Letter the truss according to Bow's Notation. Starting with the 2,000-lb wind load at the left end, lay down the known loads to scale on the load line of Fig. 103 (*b*) from A to I. The point J, which determines the amounts and directions of the reactions, can be found by the funicular polygon solution which is discussed in the solution of Fig. 102, Art. 33. A shorter solution is to combine the resultant of the wind loads and the resultant of the dead loads by the parallelogram method. This resultant reversed can then be resolved into a vertical reaction R_1 and horizontal and vertical components of the reaction R_2 (see Art. 25). With these forces known, point J in Fig. 103 is easily located. Arrowheads are placed on the load and reaction vectors.

Take the pin over the left reaction as the first free body. Going around this joint in a clockwise direction, the forces are *ja*, *ab*, *bc*, *cm*, and *mj*. The forces JA, AB, and BC are already laid down to scale in Fig. 103 (*b*). From C draw a line parallel to *cm* and through J draw a line parallel to *jm*. The intersection M of these two lines determines the stresses in *cm* and *mj*.

The diagram just completed represents the force polygon for the pin over the left reaction. The pin is in equilibrium; and, if arrows were put on the vectors, they should follow around in the usual way, back to the starting point. It is customary to place arrows on the known loads and reactions, and to omit them from the vectors representing the unknown stresses because of the confusion caused when the polygons for the succeeding joints are drawn.

The kind of stress in each member—tension or compression—is determined in the following manner. In the force polygon just

completed, the directions of the known forces JA, AB, and BC are indicated by the arrowheads in Fig. 103 (b). Since the pin over the left reaction is in equilibrium, the arrows for the vectors cm and mj would, if shown, follow around to the starting point J. In

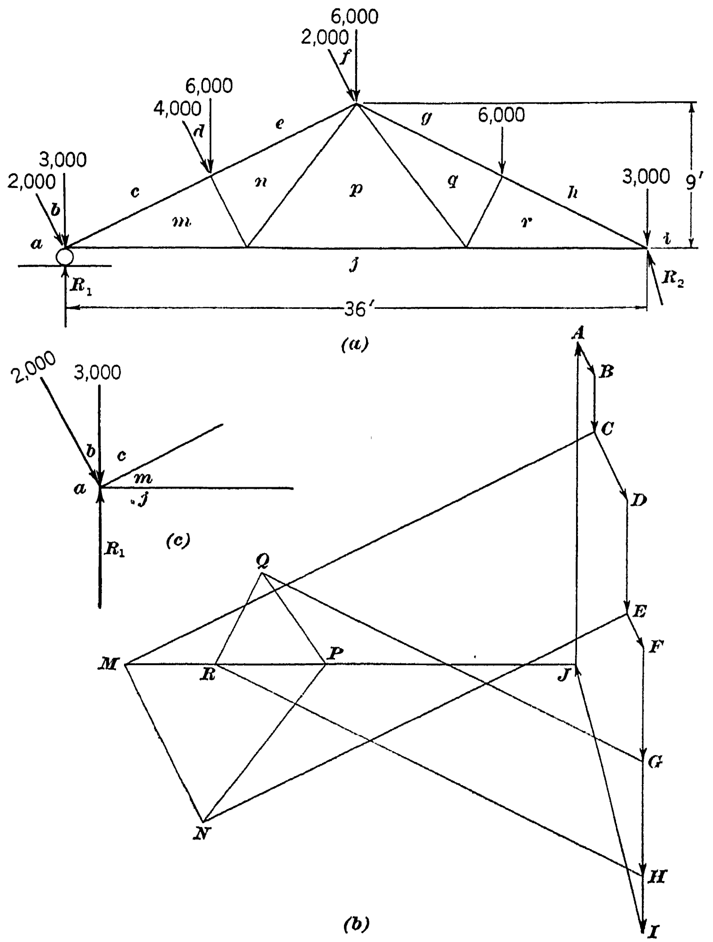


Fig. 103

order to do this, cm must act down to the left and mj must act horizontally to the right. Therefore, cm pushes on the pin at R_1, and the stress in cm is compression; mj pulls on the pin at R_1, and the stress in mj is tension.

The kind of stress in a member may also be determined in the following manner. If the letters which designate the forces acting on the pin at R_1 are read in a clockwise direction, the two unknown stresses are read as *cm* and *mj*. In going from *C* to *M* on the force diagram, Fig. 103 (*b*), the direction is down to the left or toward the pin. Member *cm* pushes on the pin; therefore, it is a compression member. In a similar manner, when going from *M* to *J* in Fig. 103 (*b*), the direction is horizontally to the right or away from the pin. The stress in *mj* is thus determined to be tension.

The next joint or pin with only two unknowns is the second pin on the top chord. In a clockwise direction, *mc* is the first known stress. Stresses *MC*, *CD*, and *DE* are already laid down on the force diagram, Fig. 103 (*b*). Through *E* draw a line parallel to *en*, and through *M* draw a line parallel to *mn*. These two lines intersect at *N*. Reading around the pin in a clockwise direction, the unknowns are designated by *en* and *nm*. The stress in *en* is compression, because in going from *E* to *N*, Fig. 103 (*b*), the direction is toward the pin; for the same reason the stress in *nm* is also compression.

The third pin is the one directly below on the lower chord. The known stresses in a clockwise direction are *jm* and *mn*. The vectors representing these stresses are drawn on the force diagram. Starting at *N*, draw a line parallel to *np* and through *J* draw a line parallel to *jp*. These intersect at *P* and determine *np* and *pj*. When going around the joint in a clockwise direction, the unknowns are designated by *np* and *pj*. From *N* to *P*, Fig. 103 (*b*), is away from the pin and thus indicates tension in *np*; from *P* to *J* is also away from the pin, and so the stress in *pj* is tension also.

The next pin with only two unknown stresses is the center top pin. This pin and the remaining joints may be solved in the manner previously explained.

PROBLEM

114. Solve for the stresses in all members of the trusses shown in Figs. 104, 105, 106, and 107.

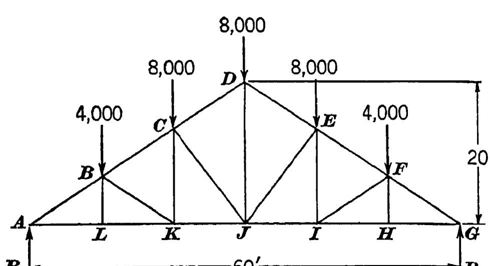


FIG. 104

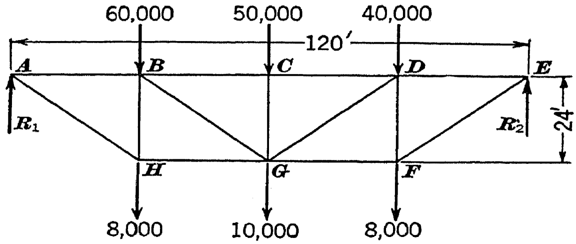


FIG. 105

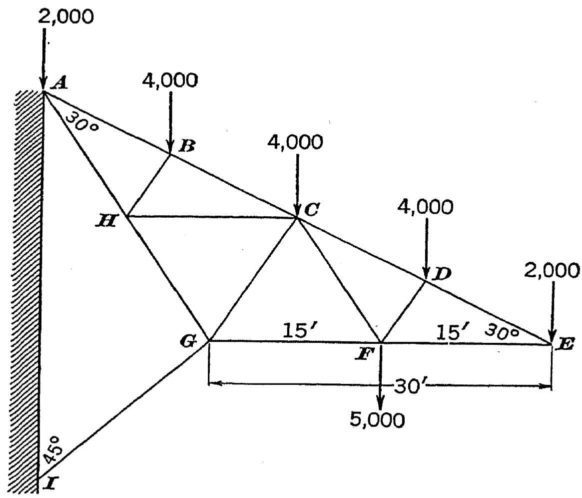


FIG. 106

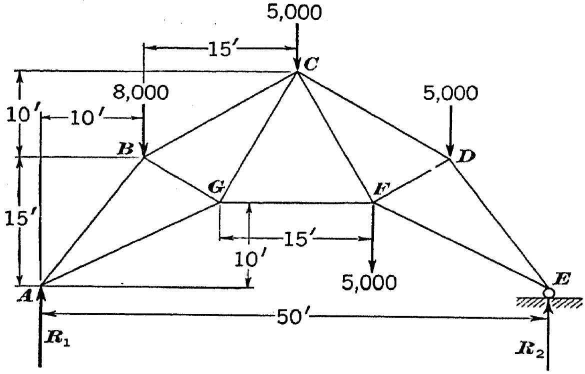


FIG. 107

36. Joints With More than Two Unknown Stresses.—There are two methods by which a joint which has more than two unknown stresses can be solved:

(*a*) By the method of sections;

(*b*) By substitution of a false member.

37. Method of Sections.—Consider the truss shown in Fig. 108 (*a*). The reactions may be solved for by the method given in Art. 33, or by inspection. Starting at the left reaction, the first three pins can be solved by the method of Art. 35. When the fourth and fifth pins are considered, it will be seen that each has three unknown stresses acting on it. The method of Art. 35 cannot be applied to these joints.

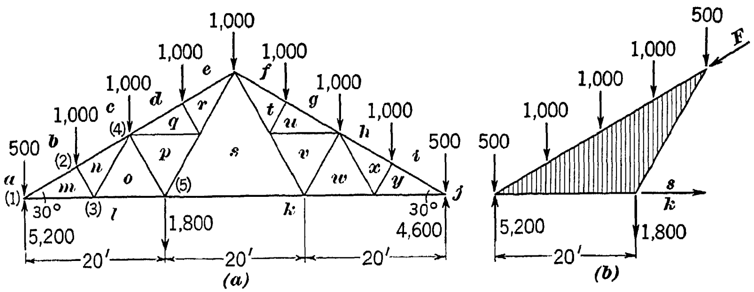


FIG. 108

Fig. 108 (*b*) shows the left half of the truss taken as a free body. This free body is held in equilibrium by seven known forces, the unknown reaction at the top center pin, and the unknown stress in the horizontal member *sk*. The known forces could be combined into a single resultant force, thereby reducing the forces acting on the free body to one known and two unknown forces. The unknowns could then be found by constructing a force triangle of the three forces. Unfortunately the resultant of the seven known forces is a force of 600 lb acting downward at a point 160 ft to the right of the left reaction. It would be practically impossible to construct a force triangle by using the 600-lb resultant and the two unknown forces.

The stress in member *sk* can be divided into two parts; a tension caused by the 5,200-lb reaction and a compression caused by the downward loads.

In Fig. 109 (*a*) the left half of the truss is shown held in equilibrium by the 5,200-lb reaction, an unknown force F at o, and an unknown tension sk. The 5,200-lb reaction, sk, and F intersect at the pin at R_1. Lay off the 5,200-lb vector upward from the pin at R_1. Through the end of this vector draw T parallel to sk and intersecting the line connecting the pin at R_1 with o. It is found that $T = 9{,}010$ lb. This is the tensile component of the stress in sk.

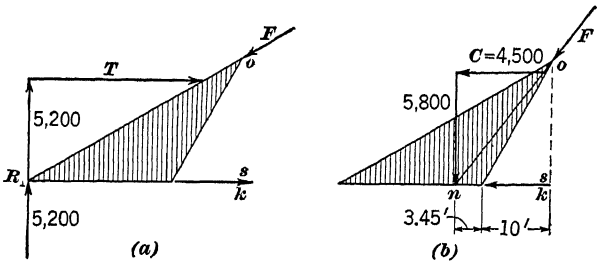


FIG. 109

Fig. 109 (*b*) shows the left half of the truss held in equilibrium by the 5,800-lb resultant of the known loads, a force at F, and a compressive stress in sk. These three forces intersect at n. From n lay off the 5,800-lb vector, and through its upper end draw C parallel to sk. The compressive stress in sk is $C = 4{,}500$ lb. The algebraic sum of the tensile and compressive stresses, or $9{,}010 - 4{,}500 = 4{,}510$, is the resultant tension in sk.

38. Method by Substitution of a False Member.—In Fig. 110 (*b*) the solution for the stresses acting at the first three joints at the left end of the truss shown in Fig. 110 (*a*) is made in the usual manner.

The truss shown in Fig. 110 (*a*) is the same as that shown in Fig. 108 (*a*), but members pq and qr have been replaced by a single member $p'r$. After this change is made, the joints *4*, *6*, and *5* can be solved by the method of joints, as shown in Fig. 110 (*b*), as they now have only two unknowns each. Points R and S can be located on the force diagram. The false member $p'r$ may now be removed, and members pq and qr may be put back. The solution can now be completed in the usual manner.

If the left half of the truss is taken as a free body, Fig. 110 (*c*), it will be observed that the stresses in the members er and rs are

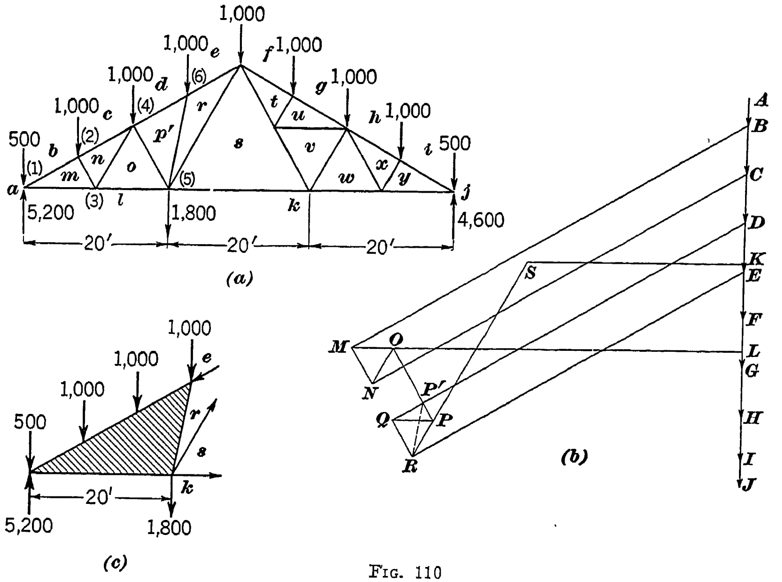


FIG. 110

not affected by any change which is made in the truss to the left of joint *6*. The positions of the points *R* and *S* on the force diagram are therefore not affected by changes in the truss to the left of joint *6*.

PROBLEMS

115. Determine the reactions and stresses in all members of the Fink truss shown in Fig. 111.

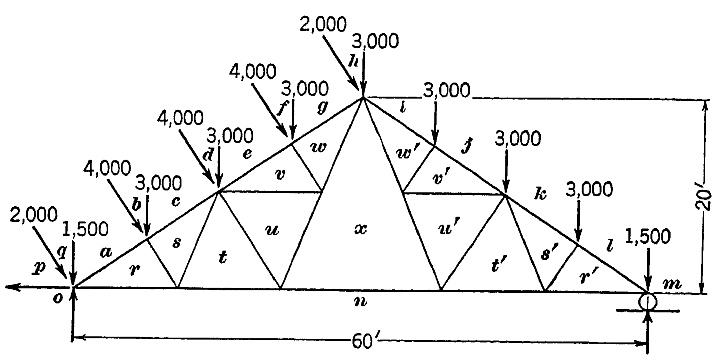


FIG. 111

116. Determine the reactions and stresses in all members of the cambered Fink truss of Fig. 112.

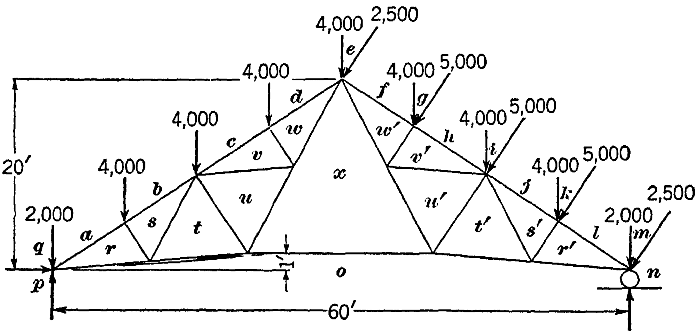


FIG. 112

39. Three-Force or Multi-Force Members.—If any member of a structure has forces acting on it at more than two points, it is called a "three-force member" or "multi-force member."

As was explained in Art. 34, in the case of roof and bridge trusses all loads and reactions are applied at the joints or pins. The members of such trusses have forces acting on them at two points only—the ends of the member. These are "two-force members" and carry direct tension or direct compression only. The effect of the weight of the member is usually neglected; or in the case of heavy compression members the weight is divided between the two end pins.

There are other structures, however, such as cranes and various types of frames, in which loads are applied to the members at one or more points between the end pins. In these multi-force members the stress condition is complex, consisting of a combination of tension and compression caused by bending, shear, and direct tensile or compressive stresses.

The effect of a two-force member on the pin at each end of the member is a *straight push or pull along the axis of the member.* Such a condition can be represented by a single force vector.

The effect of a multi-force member at any pin can best be explained by examining the action of member AB in Fig. 113 (a). This member has forces acting at A, B, and D, which cause it to bend approximately as indicated in Fig. 113 (b).

It is easily seen that, at A, the effect of the two-force member AC can be fully represented by the vector AC, but the effect of the three-force member ADB on the pin at B cannot be repre-

sented by a single vector acting along the line AB in Fig. 113 (a), because of the bending and shear which are present.

A two-force member can be cut at any point between the pins; and if two equal and opposite forces, each equal to the stress in the member, are introduced, the rest of the structure will be unaffected by the change. This cannot be done in the case of a three-force member, because the stress condition is different at each point along the member. In order to study the effect of a multi-force member on the other parts of a structure, it is necessary to remove the member, place it in equilibrium, and determine the reactions at the various pins.

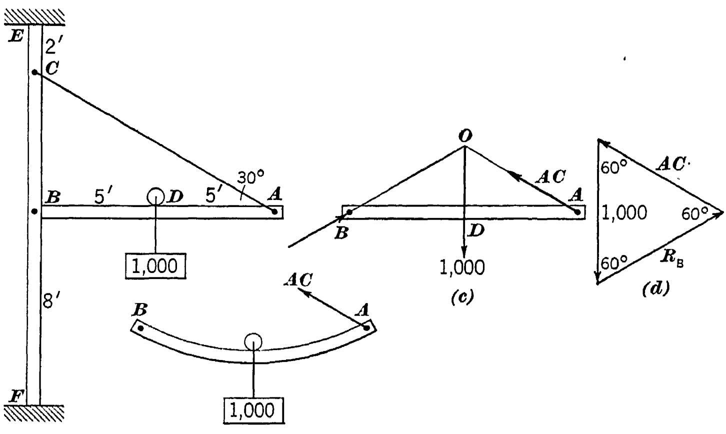


FIG. 113

The solution for the pin reactions of multi-force members will be illustrated in the following example.

EXAMPLE

For the crane shown in Fig. 113 determine the stress in AC, the pressure on the pin at B, and the reactions at the floor and ceiling.

The member AC is a two-force member, because the only forces acting on it are applied at the ends A and C. It is a tension member. The member AB is a three-force member, because it has forces acting at A, B, and D. The post is also a multi-force member, since it has forces acting at the points B, C, E, and F.

Fig. 113 (c) shows the member AB taken out as a free body. Since it is in equilibrium and is being acted on by three forces,

these three forces must meet in a common point, as is explained in Art. 20. The lines of action of the force AC and the 1,000-lb weight intersect at the point O; therefore, the direction of the pin reaction at B must be along the line BO. The force triangle for these three forces is shown in Fig. 113 (d). Since all angles of the triangle are 60°, the tension in AC and the pin reaction at B are each equal to 1,000 lb.

The post of the crane is shown as a free body in Fig. 114 (a). Produce the forces acting at B and C until they intersect at O. Construct the parallelogram and determine the resultant S, which is also 1,000 lb. The post is now held in equilibrium by S, the horizontal force R_1 at E, and a force R_2 acting at F. The forces R_1 and S intersect at t. The reaction at R_2 must also pass through t. Fig. 114 (b) is the force triangle which determines R_1 and R_2. The reactions R_1 and R_2 could also have been obtained from a similar solution applied directly to the entire crane, Fig. 113 (a), as a free body. It is found that

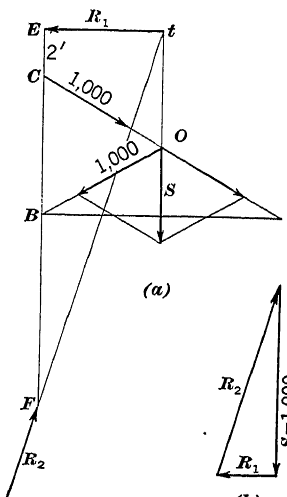


FIG. 114

$$R_1 = 317 \text{ lb and } R_2 = 1{,}048 \text{ lb}$$

PROBLEMS

117. Determine the stress in member CB, the pin pressure at D, and the reactions at E and F in Fig. 115. *Ans.* $BC = 2{,}690$ *lb, T.; 2,413 lb; 707 lb; 1,225 lb.*

118. Solve for the stress in CD, the pin pressure at B, and the reactions at E and F, Fig. 116. Resolve the reaction at F into horizontal and vertical components. Explain the relationship between the reaction at E and the horizontal component of the reaction at F.

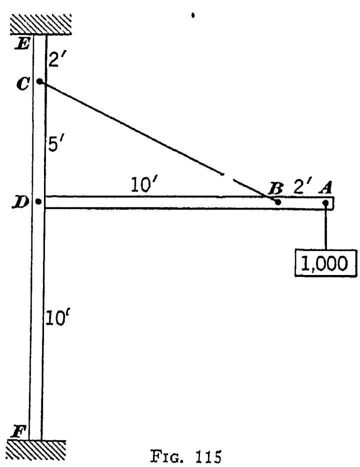


FIG. 115

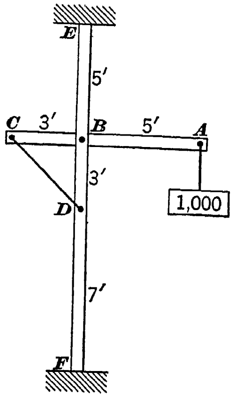


FIG. 116

119. Determine graphically the stress in the cord AB and the pin reaction at C in Fig. 117.

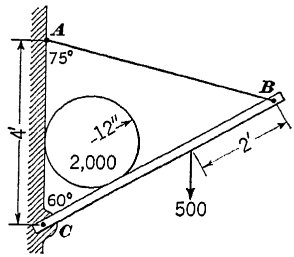


FIG. 117

40. Bents.—A bent is a roof truss supported on two columns and braced with knee-braces. This method of support is quite different from resting the truss on top of two walls. The columns are either hinged at the ground or rigidly fixed by bolts or other means.

The wind pressure on the side and roof of the building causes a side thrust, which develops a bending action in the supporting columns. Because of this bending action, the method of joints cannot be applied to the columns of a bent. The columns of a bent are three-force members; therefore, each column must be taken out as a whole and treated as a free body in equilibrium.

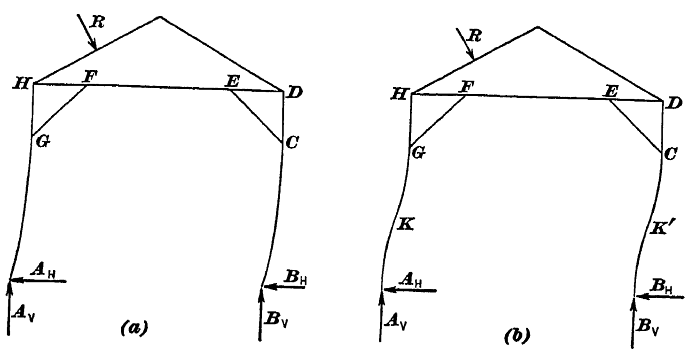


FIG. 118

Fig. 118 (a) shows the distortion of a knee-braced bent, hinged at the column bases, when subjected to a resultant diagonal load such as is produced by wind and dead loads. The same bent is

shown in Fig. 118 (*b*), but in this case the column bases are rigidly fixed. This change in method of attachment changes the curvature of each column, and produces points of inflection[1] at K and K'. Since hinges could be placed at K and K', the net result is that the effective height of the supporting columns of the bent is reduced. This reduces the vertical components of the reactions at the bases of the columns. Reduction of the reactions produces, in general, a reduction in stress in the various members of the bent.

Since it is difficult to produce an absolutely rigid base connection, most so-called fixed-base columns are probably somewhere between the fixed end condition and the condition produced by hinged ends. The hinged end condition produces the largest stresses and therefore will be the condition assumed in this book. For the study of fixed end conditions the student is referred to the standard texts on structures.

The division of the horizontal thrust, which is caused by the wind load, cannot be exactly determined. It is dependent on the rigidity of the truss and the relative rigidity and size of the columns. If both columns are of the same size and all other conditions are perfect, it would be reasonable to assume that each column would take half of the horizontal wind thrust. It is safer, however, to assume that all the horizontal thrust is resisted by one column, and that the reaction at the base of the other column is vertical. This assumption leads to greater stresses in the members of the bent.

EXAMPLE

Solve for the reactions and the stresses in all members of the bent shown in Fig. 119 (*a*). Assume that the horizontal thrust is equally divided between the reactions.

Collect all vertical loads into their resultant, $R_3 = 16{,}000$ lb. The resultant of the wind loads is $R_4 = 4{,}000$ lb. Produce R_3 and R_4 until they intersect. The resultant of R_3 and R_4 is R_5, and the resultant horizontal wind thrust is $R_6 = 10{,}000$ lb. Com-

[1] A point of inflection in a beam or column is a point where the curvature of the member, due to bending action, changes from convex to concave. At such a point there is no bending stress set up in the member. The stress in the member is either a pull or a push along the axis of the piece. Under this condition of stress it would be possible to place a hinge at the point of inflection without destroying the equilibrium.

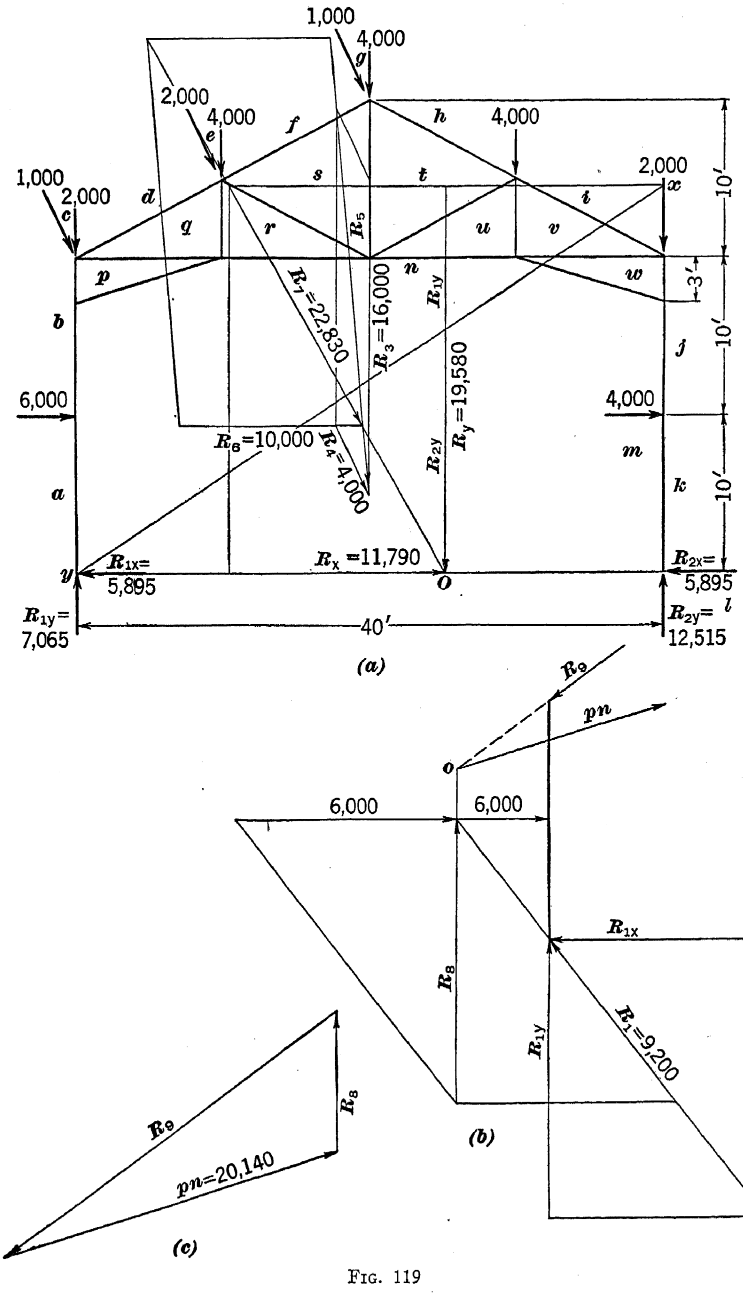

1,000
4,000
2,000
4,000
4,000
1,000
2,000
2,000
6,000
4,000
R_7=22,830
R_3=16,000
R_y=19,580
R_6=10,000
R_4=4,000
R_5
R_{1y}
R_{2y}
R_x =11,790
R_{1x}=
5,895
R_{2x}=
5,895
R_{1y}=
7,065
R_{2y}=
12,515
40′
10′
3′
10′
10′
(a)
R_9
pn
6,000
6,000
R_8
R_{1x}
R_{1y}
R_1=9,200
(b)
R_9
R_8
pn=20,140
(c)

FIG. 119

bine R_5 and R_6 to obtain $R_7 = 22{,}830$ lb, which is the resultant of all loads acting on the bent. Produce R_7 until it intersects the line through the pins at the bases of the columns. At point O, resolve R_7 into its horizontal and vertical components: $R_x = 11{,}790$ lb; $R_y = 19{,}580$ lb. These components act at point O. The horizontal thrust R_x is divided equally between the two hinges; thus, $R_x \div 2 = R_{1x} = R_{2x} = 5{,}895$ lb.

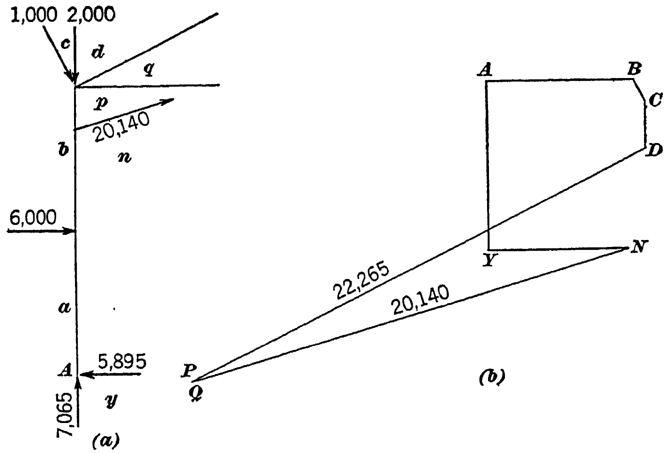


FIG. 120

The next step is to divide the vertical component of R_7, or R_y, into the two forces which act at the right and left reactions. Through the upper end of R_y draw a line parallel to the base line and intersecting the right column at point x. Connect the point x with the pin at the left reaction. This diagonal line will divide R_y by inverse proportion into $R_{1y} = 7{,}065$ lb and $R_{2y} = 12{,}515$ lb. These forces are the vertical components of the reactions at the bases of the columns.

The left column is shown as a free body in Fig. 119 (b). Combine R_{1x} and R_{1y} into their resultant $R_1 = 9{,}200$ lb. Produce R_1 until it intersects the 6,000-lb wind pressure. Combine the 6,000-lb force and R_1 into their resultant R_8. The column is now held in equilibrium by R_8, the stress in the knee-brace pn, and the four forces acting at the top of the column. R_8 and pn intersect at point o. The resultant of the four forces acting at the top of the column must also pass through this point o and the pin at the top of the column. Fig. 119 (c) shows the force triangle for the

three forces intersecting at the point o in Fig. 119 (b). The stress in the knee-brace pn is found to be 20,140 lb, tension.

In Fig. 120 (a) the left column is shown as a free body with the four original forces acting at the top, the known stress pn, the wind pressure of 6,000 lb, and the two components of the reaction at the pin A at the base of the column. Since this free body has only two unknown forces acting on it, these may be solved for by drawing a force polygon as shown in Fig. 120 (b).

Some work may be saved by laying down the load line for the entire bent and attaching the force polygon of Fig. 120 (b) to this load line.

The remaining internal stresses of the bent may be found by the method of joints, the load line being used as a base for the various force polygons.

PROBLEMS

120. In the bent shown in Fig. 121, the left reaction is assumed to be vertical and the right hinged. Determine the reactions and the stresses in all members of the bent. *Ans.* $R_1 = 13{,}250$ *lb*; $R_{2V} = 16{,}750$ *lb*; $R_{2H} = 6{,}000$ *lb*; $d = 6{,}750$ *lb*, T.

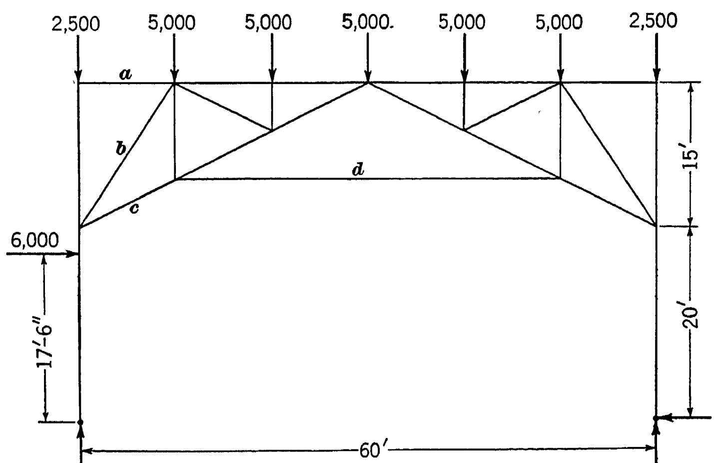


FIG. 121

121. The columns of the bent shown in Fig. 122 are hinged. Assume that the left reaction takes all of the horizontal thrust. Solve for the reactions and the stresses in all the members of the bent.

122. Determine the stresses in all members of the airplane-engine nacelle shown in Fig. 123.

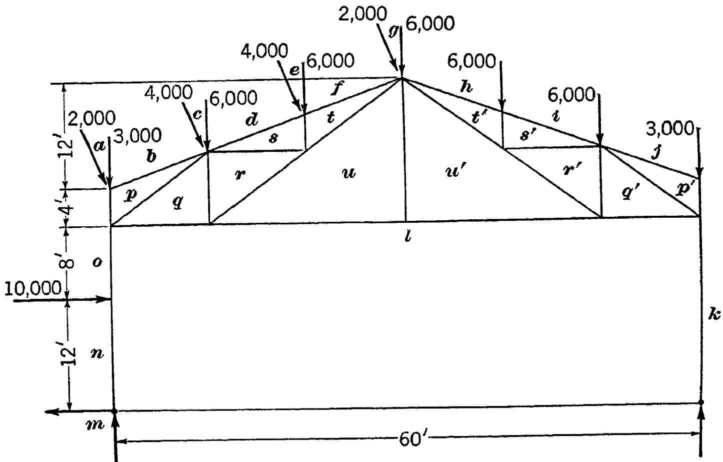


FIG. 122

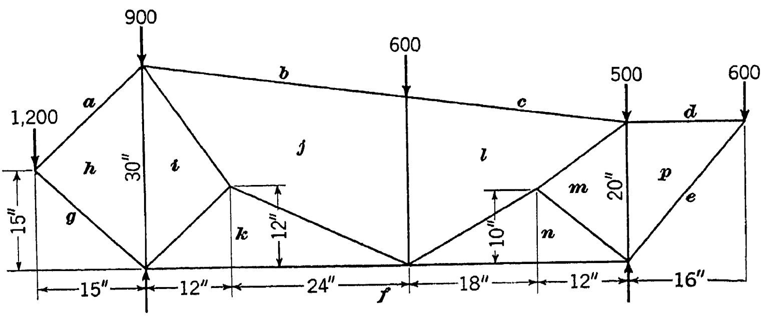


FIG. 123

CHAPTER 5

COPLANAR, NON-CONCURRENT FORCE SYSTEMS BY MATHEMATICAL METHODS

41. **Review of Definitions.**—A coplanar, non-concurrent force system consists of several forces, all of which have their lines of action in a common plane but which do not pass through a common point.

The resultant of a coplanar, non-concurrent force system is the single force or couple which will produce the same effect as the several forces acting together.

42. **Resultant of Coplanar, Non-Concurrent Force Systems.** To determine the resultant of a system of coplanar, non-concurrent forces by the mathematical method, the first step is to resolve each of the forces of the system into components parallel to any two intersecting axes which lie in the plane of the forces. It is usually preferable to resolve the forces into horizontal and vertical components or into components which are parallel to axes which meet at 90°. These components are added vectorially, and their resultant is given by the equation $R=\sqrt{(\Sigma F_x)^2+(\Sigma F_y)^2}$. The angle which R makes with the X axis is given by $\tan\theta=\frac{\Sigma F_y}{\Sigma F_x}$.

The above solution gives the amount and direction of the resultant, but not its position. By the principle of moments, Art. 16, *the moment of the resultant with respect to any axis perpendicular to the plane of the forces is equal to the algebraic sum of the moments of the component forces, with respect to the same axis.* Thus, if r is the perpendicular distance to the resultant force R from an axis through any point O in the plane of the forces, and d_1, d_2, and d_3 are the perpendicular distances to the forces F_1, F_2, and F_3 from the same axis, the position of the resultant force R will be determined by the equation

$$\Sigma M_O=Rr=F_1d_1+F_2d_2+F_3d_3$$

If both $\Sigma F_x=0$ and $\Sigma F_y=0$, but ΣM_O is not zero in the foregoing discussion, then the resultant force is zero and the resultant

of the force system is a couple, the magnitude and sense of whose moment is determined by the value of ΣM_O.

EXAMPLE

Determine the amount and position of the resultant of the system of forces shown in Fig. 124.

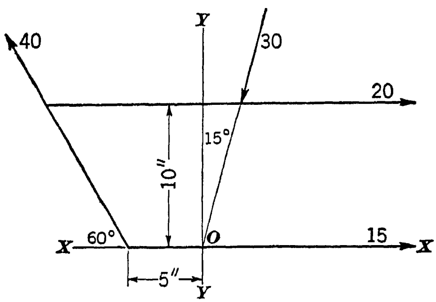


FIG. 124

Let the X and Y axes be as shown. Resolve each force into its X and Y components.

$$\Sigma F_x = 15 + 20 - 7.76 - 20 = 7.24 \text{ lb}$$
$$\Sigma F_y = -29 + 34.65 = 5.65 \text{ lb}$$
$$R = \sqrt{7.24^2 + 5.65^2} = 9.2 \text{ lb}$$
$$\tan\theta = \frac{5.65}{7.24} = 0.78$$
$$\theta = 38° \text{ with the } X \text{ axis}$$
$$\Sigma M_O = -9.2\,r = -20 \times 10 - 40 \times 0.866 \times 5$$
$$r = 40.5 \text{ in.}$$

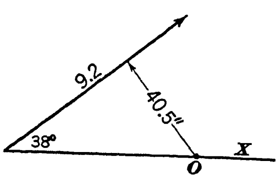


FIG. 125

Therefore, as indicated in Fig. 125, the resultant R is a force of 9.2 lb acting to the right at 38° above the X axis, at a perpendicular distance $r = 40.5$ in. from O, and in such a sense that rotation is in a clockwise direction about O.

PROBLEMS

123. In Fig. 124 change the 30-lb force to a 50-lb force, and let the 15-lb force be replaced by a 25-lb force acting to the left. Determine the amount and position of the resultant force. *Ans.* *40.3 lb; 199.8°; 9.25 in.*

124. Determine the amount and position of the resultant of the force system shown in Fig. 126.

125. Solve for the amount and position of the resultant of the forces shown in Fig. 127.

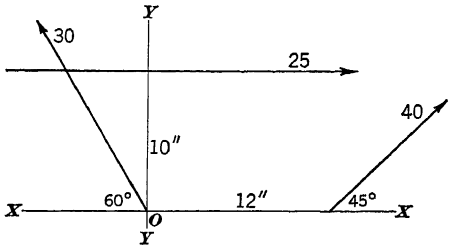


FIG. 126

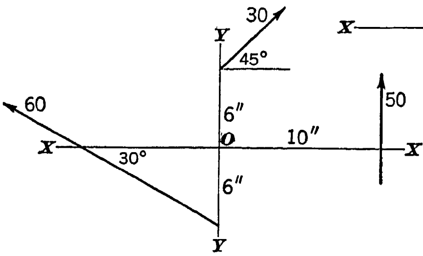


FIG. 127

126. Determine the magnitude, direction, and position of the resultant of the forces in Fig. 128.

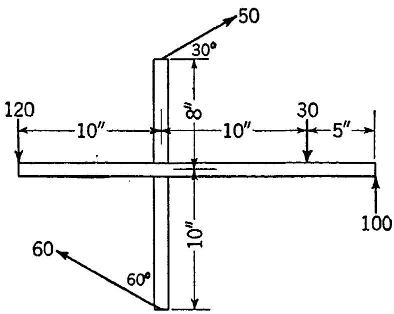


FIG. 128

43. Equilibrium of Coplanar, Non-Concurrent Systems. For equilibrium of a coplanar, non-concurrent force system, there must be no acceleration along either of any two intersecting lines in the plane of the forces. Thus, $\Sigma F_x=0$ and $\Sigma F_y=0$. There must also be no rotation about any axis perpendicular to the plane of the forces, or $\Sigma M=0$.

Since three conditions must be satisfied for equilibrium, three independent equations can be written; therefore, three unknown quantities can be solved for. Thus,

$$\Sigma F_x=0;\ \Sigma F_y=0;\ \Sigma M=0.$$

EXAMPLE 1

Fig. 129 represents a bell-crank lever with a bearing at B. Determine the force P and the reaction at B necessary for equilibrium.

Since the direction of the reaction at B is unknown, it is represented by its horizontal and vertical components, B_x and B_y. The system then has three unknowns, the magnitudes of three forces.

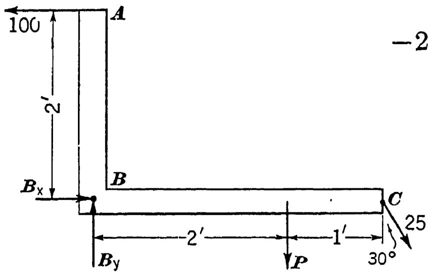


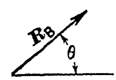
FIG. 129

$$\Sigma M_B=0$$
$$-2P-25\times0.866\times3+100\times2=0$$
$$P=67.5\text{ lb}$$

$$\Sigma F_x=0$$
$$-100+B_x+25\times0.5=0$$
$$B_x=87.5\text{ lb}\rightarrow$$

$$\Sigma F_y=0$$
$$B_y-67.5-25\times0.866=0$$
$$B_y=89.15\text{ lb}\uparrow$$

$$R_B=\sqrt{87.5^2+89.15^2}=124.7\text{ lb}$$
$$\tan\theta=\frac{89.1}{87.5}=1.02$$
$$\theta=45.5°$$



EXAMPLE 2

Determine the pressure on the roller at C, and the amount and direction of the pull required at A, for equilibrium of the triangular block shown in Fig. 130 (a)

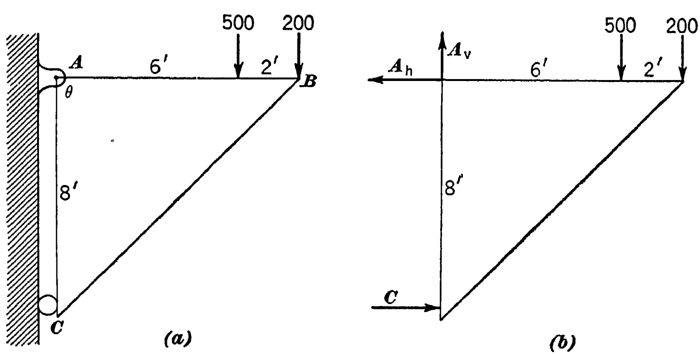


FIG. 130

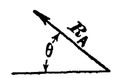
In Fig. 130 (b) the block is shown as a free body, with the pull at A represented by its horizontal and vertical components.

$$\Sigma M_A = 0$$
$$8C - 500 \times 6 - 200 \times 8 = 0$$
$$C = 575 \text{ lb} \rightarrow$$

$$\Sigma F_h = 0$$
$$A_h = C = 575 \text{ lb} \leftarrow$$

$$\Sigma F_v = 0$$
$$A_v - 500 - 200 = 0$$
$$A_v = 700 \text{ lb} \uparrow$$

$$R_A = \sqrt{575^2 + 700^2} = 905 \text{ lb}$$
$$\tan\theta = \frac{700}{575} = 1.218$$
$$\theta = 50.6°$$



PROBLEMS

127. Fig. 131 represents a beam hinged at A and supported by a roller at B. Compute the amounts of the reactions necessary at A and B for equilibrium. *Ans.* $A_x = 91.4$ *lb;* $A_y = 128.3$ *lb;* $B_y = 113.1$ *lb;* $A = 157$ *lb; 125.45°.*

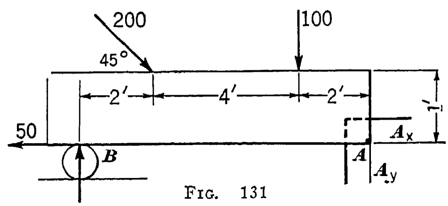


FIG. 131

128. The flywheel shown in Fig. 132 is acted upon by two belt pulls and the thrust P from the connecting-rod. Determine the amount of thrust P necessary for a constant rotative speed, and also the bearing reaction at A when P and the belt pulls are acting.

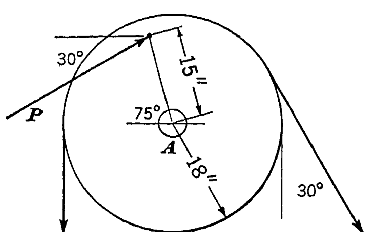


FIG. 132

129. A 20-ft ladder weighing 70 lb rests against a smooth wall at an angle of 30° with the wall. A 200-lb man climbs to within 5 ft of the top. What are the pressure on the wall and the amount of the reaction at the base of the ladder?

130. Determine R_1 and R_2 for the truss of Fig. 133.

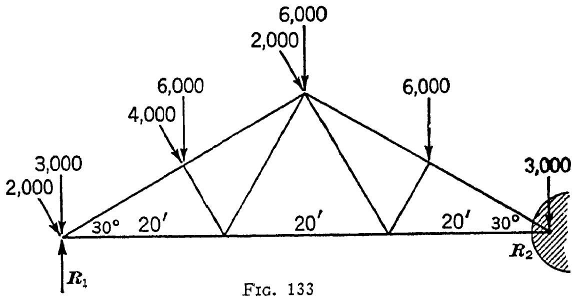


FIG. 133

131. Fig. 134 represents a 500-lb door, hinged at A. Determine the reaction at A and the force P necessary to maintain the 30° position. *Ans. 762 lb; 577 lb; 40.9°.*

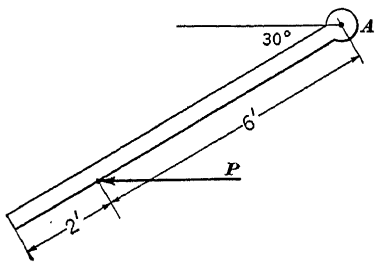


FIG. 134

132. A cantilever truss, Fig. 135, is hinged at A and held out from the wall by the strut BC. All joints are pin connections. Compute the reaction at A and the compression in the strut BC.

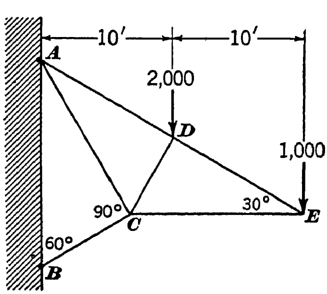


FIG. 135

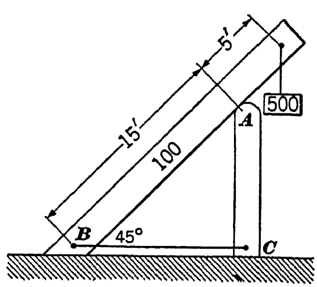


FIG. 136

133. The bar shown in Fig. 136 is resting against smooth surfaces at points A and B. The bar weighs 100 lb. Compute the forces acting at A and B, and also the tension in the rope BC.

44. **Trusses.**—A truss is a structure made up of straight bars joined together at the ends by pins in such a manner that the bars form triangles. For a discussion of the methods of loading and stress action, the student is referred to Art. 34.

45. **Stresses in Trusses by Method of Joints.**—Each joint or pin of a truss is acted upon by a coplanar, concurrent force system. The conditions to be satisfied for equilibrium of such a system are two, namely, $\Sigma F_x=0$ and $\Sigma F_y=0$.

Since only two independent equations can be written, the unknown forces at any pin cannot exceed two, if a solution is to be obtained by the method of joints.

The procedure for solving a truss by the method of joints is as follows:

(1) Using the entire truss as a free body, solve for the external reactions by applying the moment equation $\Sigma M=0$ with respect to two different axes. Check the results by vertical and horizontal summation of forces, or by use of the equations $\Sigma F_x=0$ and $\Sigma F_y=0$.

The reactions for some trusses may be more easily determined by inspection and the principle of inverse proportion. Reactions obtained in this manner should also be checked by the equations $\Sigma F_x=0$ and $\Sigma F_y=0$.

(2) Select a pin at which *not more than two unknown forces are acting*—usually the pin directly over one of the reactions. An exception is the cantilever truss, where the first pin selected generally is the pin at the free or unsupported end of the truss. The selected pin is isolated as a free body. The coplanar, concurrent force system consisting of the known forces and *not more than two unknown stresses* is solved by one of the methods developed in Chapter 2.

(3) Select the next pin which has no more than two unknown stresses. Isolate this pin as a free body and solve as before. Continue this process until all unknown stresses are determined.

EXAMPLE 1

Solve for the reactions and stresses in all members of the truss shown in Fig. 137 (*a*).

Since the loads are symmetrically placed, it is evident that each of the reactions must be 3,000 lb.

Fig. 137 (*b*) shows the pin at A as a free body. This pin is acted upon by the known reaction of 3,000 lb and the unknown stresses in members AB and AD. These unknown stresses become external forces when the pin is isolated as a free body.

$$\Sigma F_v = 0 \qquad\qquad \Sigma F_h = 0$$
$$3{,}000 - AB \sin 30° = 0 \qquad\qquad AD - AB \cos 30° = 0$$
$$AB = 6{,}000 \text{ lb, C.} \qquad\qquad AD = 5{,}196 \text{ lb, T.}$$

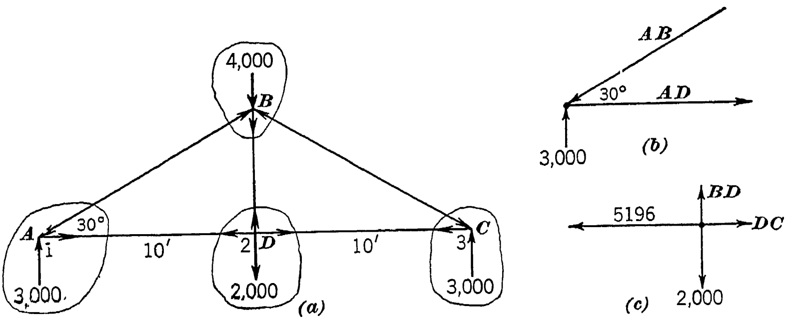


FIG. 137

The next pin to be considered is the one at D. This pin is shown as a free body in Fig. 137 (*c*). The pin is acted upon by the known tension of 5,196 lb in AD, the 2,000-lb external load, and the two unknown stresses in members BD and DC.

$$\Sigma F_v = 0 \qquad\qquad \Sigma F_h = 0$$
$$BD - 2{,}000 = 0 \qquad\qquad DC - 5{,}196 = 0$$
$$BD = 2{,}000 \text{ lb, T.} \qquad\qquad DC = 5{,}196 \text{ lb, T.}$$

The only remaining unknown stress is BC. Since the truss is symmetrically loaded, the stress in BC must be equal to that in AB or 6,000 lb, compression.

EXAMPLE 2

Solve for the reactions and the stresses in all members of the truss shown in Fig. 138.

$$\Sigma M_D = 0 \qquad\qquad \Sigma M_A = 0$$
$$6{,}000 \times 15 + 3{,}000 \times 30 \qquad\qquad 45\, R_2 - 15 \times 3{,}000 -$$
$$-45\, R_1 = 0 \qquad\qquad 30 \times 6{,}000 = 0$$
$$R_1 = 4{,}000 \text{ lb} \qquad\qquad R_2 = 5{,}000 \text{ lb}$$

Check: $\Sigma F_v = 0$, or $4{,}000 + 5{,}000 - 3{,}000 - 6{,}000 = 0.$

The reactions may also be determined by inspection, for by inverse proportion two-thirds of the 3,000-lb load, or 2,000 lb, is

carried by R_1 and one-third, or 1,000 lb, is carried by R_2. In the case of the 6,000-lb load, one-third, or 2,000 lb, is carried by R_1 and two-thirds, or 4,000 lb, by R_2. Thus, R_1 is 4,000 lb and R_2 is 5,000 lb.

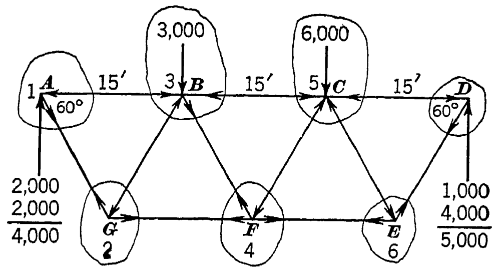


FIG. 138

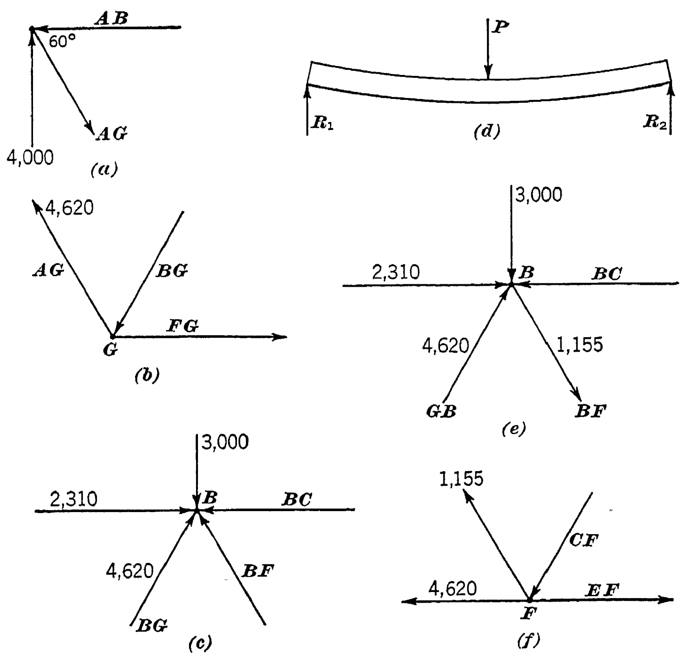


FIG. 139

In Fig. 139 (*a*), the pin over the left reaction is shown as a free body. The stresses in AB and AG become external forces when the pin is isolated.

$$\Sigma F_v = 0 \qquad \Sigma F_h = 0$$
$$4{,}000 - AG \times 0.866 = 0 \qquad 4{,}620 \times 0.5 - AB = 0$$
$$AG = 4{,}620 \text{ lb, T.} \qquad AB = 2{,}310 \text{ lb, C.}$$

At most pins it is possible to determine the direction or kind of stress in a member by inspection. At pin A the reaction of 4,000 lb is upward. Therefore, there must be a downward force to balance it or to maintain equilibrium. The member AB is horizontal and so has no vertical component. Thus, the vertical component of AG must oppose the 4,000-lb thrust of the reaction. To accomplish this, AG must pull on the pin or be a tension member. The stress in AG has a horizontal component to the right; and, since the stress in AB is the only other horizontal force, it must balance the horizontal component of AG. Since the stress in AB acts to the left, it is a compression member.

The next pin to be solved is the one at G. It is shown as a free body in Fig. 139 (*b*). Since AG is a tension member, it pulls on pin G. *Any truss supported at the ends may be thought of as a simple beam supported at the ends.* Fig. 139 (*d*) shows a beam so supported. The bottom fibers of this beam will be stretched, and those at the top will be compressed. In a like manner the bottom chords or members of a truss supported at the ends are in tension, while those at the top are in compression. By this analogy the member FG is in tension. Since AG and FG are both tension members pulling on pin G, member BG must push down to the left to maintain equilibrium. Thus, BG is a compression member.

The free body of Fig. 139 (*b*) may be solved by writing horizontal and vertical summation equations, as was done for the joint already solved; but, since the three forces acting at G are separated by angles of 60°, a force triangle of these three forces would be an equiangular triangle. The three forces must therefore be of the same magnitude, or 4,620 lb. Hence, $BG = 4{,}620$ lb, compression, and $FG = 4{,}620$ lb, tension.

Fig. 139 (*c*) shows the next free body, the pin at B. The amounts and directions of the stresses in AB and BG are now known. At a joint where four or more forces act, it is often difficult to determine the kind of stress in an unknown member by inspection. A better method is to assume a direction for each unknown, and to place arrows on the free body in accordance with the assumption made. When the equations are set up, each

unknown is given the sign it would have if the stress in the member was as shown on the diagram. The equations are then solved. If the answer has a positive sign, the assumption as to direction of the stress is correct. If the answer has a negative sign, the original assumption of direction of stress is incorrect, and the arrow indicating the direction of stress should then be changed so that it indicates the correct condition of stress. The pin at B will now be analyzed.

From Fig. 139 (c),

$$\Sigma F_v = 0$$
$$4{,}620 \sin 60° - 3{,}000 + BF \sin 60° = 0$$
$$4{,}000 - 3{,}000 + 0.866\ BF = 0$$
$$BF = -1{,}155 \text{ lb}$$

Since the answer has a negative sign, it indicates that the assumption that BF acts toward the pin, as shown in Fig. 139 (c), is incorrect. Therefore, BF is 1,155 lb, tension. Fig. 139 (e) shows the pin B with BF acting in the correct direction. From Fig. 139 (e),

$$\Sigma F_h = 0$$
$$2{,}310 + 4{,}620 \cos 60° + 1{,}155 \cos 60° - BC = 0$$
$$BC = 5{,}195 \text{ lb, C.}$$

Since the answer to the above equation is positive in sign, the original assumption as to kind of stress was correct.

The next pin with only two unknown forces acting on it is the joint F. The free body for this joint is shown in Fig. 139 (f).

$$\Sigma F_v = 0$$
$$1{,}155 \times 0.866 - 0.866\ CF = 0$$
$$CF = 1{,}155 \text{ lb, C.}$$

$$\Sigma F_h = 0$$
$$EF - 1{,}155 \times 0.5 - 1{,}155 \times 0.5 - 4{,}620 = 0$$
$$EF = 5{,}775 \text{ lb, T.}$$

The remaining unknown stresses can be computed by solving the joint C and either joint E or joint D. It is found that $CE = 5{,}775$ lb, C.; $CD = 2{,}887$ lb, C.; and $DE = 5{,}775$ lb, T.

EXAMPLE 3

Determine the reactions and the stresses in all members of the cantilever truss shown in Fig. 140 (a).

In trusses of this type it is possible to solve for the internal stresses without first determining the reactions. The pin at A

has only two unknowns and so can be used as a starting point. However, the reactions at C and D will be determined first, as this is the better method of procedure.

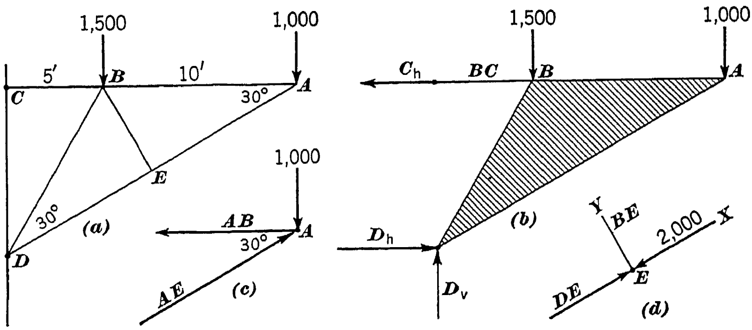


FIG. 140

In Fig. 140 (b) the whole truss is shown as a free body. The member BC could be replaced by a cord or rope, and the truss would retain its original position. Since a rope or cord can carry only tension, the direction of the reaction at C must be along the line of the pull in BC, or horizontal. If the reaction at C is horizontal, the reaction at D must have both horizontal and vertical components, in order that equilibrium may be maintained. Thus, as far as the external forces are concerned, the truss may be considered simply as a rigid triangular block which is held in equilibrium by two known forces and three unknown forces, as indicated in Fig. 140 (b).

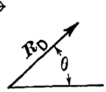
If an axis through the point D is selected as an axis of moments, the unknown forces D_h and D_v are eliminated, since they pass through D and therefore can produce no rotation about an axis through D.

$$\Sigma M_D = 0$$
$$C_h \times 10 \times 0.866 - 1{,}500 \times 5 - 1{,}000 \times 15 = 0$$
$$C_h = 2{,}595 \text{ lb} \leftarrow$$

$$\Sigma F_h = 0$$
$$D_h = C_h = 2{,}595 \text{ lb} \rightarrow$$

$$\Sigma F_v = 0$$
$$D_v - 1{,}500 - 1{,}000 = 0$$
$$D_v = 2{,}500 \text{ lb} \uparrow$$

$$R_D = \sqrt{2{,}500^2 + 2{,}595^2} = 3{,}600 \text{ lb}$$



$$\tan\theta = \frac{2{,}500}{2{,}595} = 0.962$$
$$\theta = 43.9°$$

The free body for pin A is shown in Fig. 140 (c).

$$\Sigma F_v = 0 \qquad \Sigma F_h = 0$$
$$AE \times 0.5 - 1{,}000 = 0 \qquad 2{,}000 \times 0.866 - AB = 0$$
$$AE = 2{,}000 \text{ lb, C.} \qquad AB = 1{,}730 \text{ lb, T.}$$

The free body for pin E is shown in Fig. 140 (d).

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$
$$DE - 2{,}000 = 0 \qquad BE = 0$$
$$DE = 2{,}000 \text{ lb, C.}$$

The student should study the joint at E carefully. When the free body for a joint shows members perpendicular to each other, as at E, the forces acting along the same straight line, as DE and AE, are equal and opposite. Since there is no force opposite BE, and neither AE nor DE can have a component parallel to BE, the stress in BE must be zero.[1]

Since BE is zero, BD can be easily found by treating pin B as a free body and summing forces vertically. Thus, $BD = 1{,}732$ lb, C.

Also, $BC = C_h = 2{,}595$ lb, T.

PROBLEMS

134. Solve for the reactions on the truss shown in Fig. 141. Determine the stress in each member of the truss by the method of joints. *Ans.* *AB = 6,350 lb, C.; BC = 5,200 lb, C.; BE = 4,035 lb, T.; AE = 3,175 lb, T.; CE = 2,890 lb, T.; CD = 7,500 lb, C.; DE = 3,750 lb, T.*

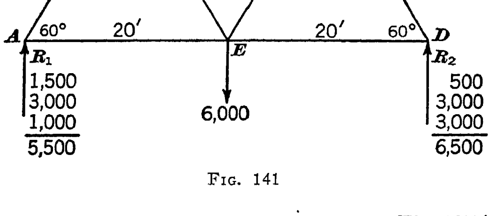


FIG. 141

[1] The student should now re-examine the joint at D, Fig. 137 (a). He should be sure that he understands how the pin at D differs from the pin at E, Fig. 140.

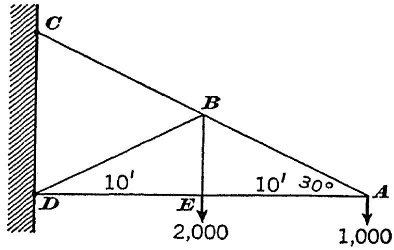


FIG. 142

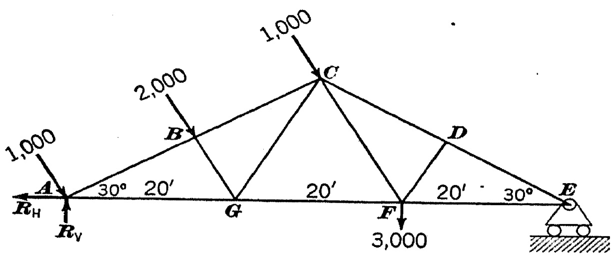


FIG. 143

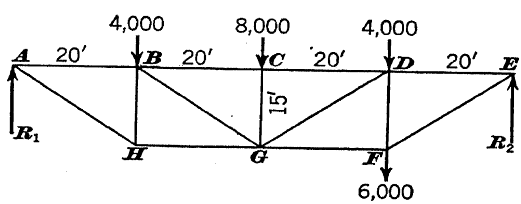


FIG. 144

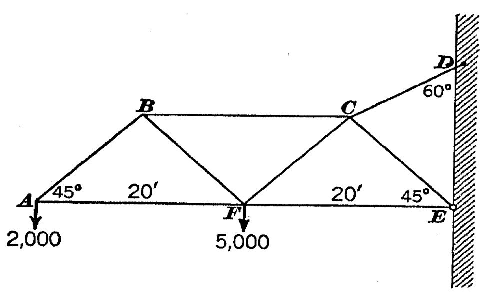


FIG. 145

135. Solve for all internal stresses and the pin pressures at pins C and D, Fig. 142.

136. The truss shown in Fig. 143 has a fixed pin support at A and rollers at E. Solve for the reactions and stresses in all members.

137. Determine the reactions by inverse proportion and solve for stresses in all members of the truss in Fig. 144.

138. Solve for the stresses in all members of the truss in Fig. 145.

139. Determine the resultant reactions at the pins at C and D and the stresses in all members of Fig. 146.

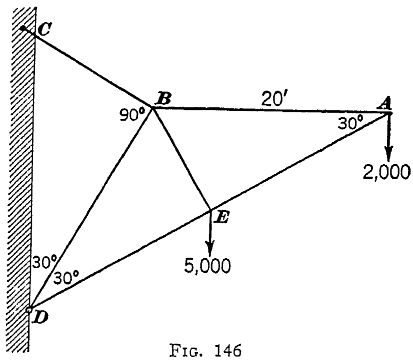


FIG. 146

46. Trusses: Solution by the Method of Sections.—Very often the stresses in certain particular members of a truss are desired. These can be obtained by working through the truss by the method given in the previous article, but particular stresses can usually be obtained with less labor by the method of sections.

The procedure for solution by the method of sections is as follows:

(1) Solve for the reactions, as in the method of joints.

(2) Pass a plane through the truss, dividing the truss into two parts *but cutting not more than three members whose stresses are unknown.* The forces acting on each part of the truss will then constitute a coplanar, non-concurrent force system. The conditions for equilibrium of such a system are $\Sigma F_x=0$; $\Sigma F_y=0$; and $\Sigma M=0$. Three independent equations can be written, and *the free body therefore cannot have more than three unknown stresses.*

(3) Draw a free body for the part of the truss acted upon by the smallest number of forces.

(4) With an axis through the point of intersection of two of the unknown stresses as the axis of moments, write a moment equation and solve for the third unknown stress. This process

may be repeated until all three unknowns are determined, or the other two unknowns may be found by using the summation equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, whichever method proves more convenient.

(5) For a truss similar to that shown in Fig. 144, with parallel top and bottom, the stress in the diagonals such as BG and DG should always be obtained from $\Sigma F_v = 0$.

EXAMPLE 1

Solve for the stresses in members BC, CF, and FE of the truss shown in Fig. 147 (*a*).

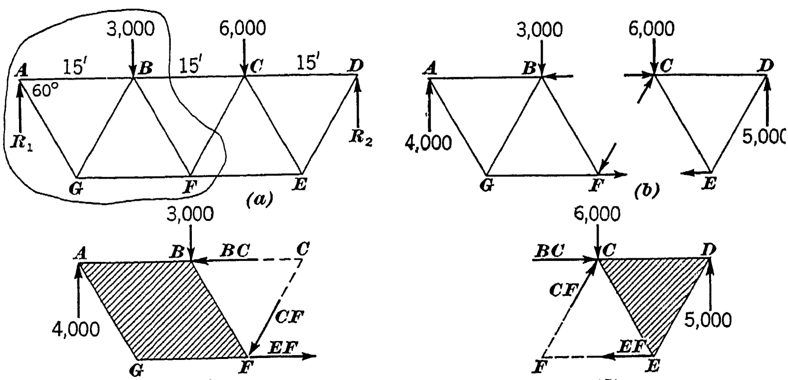


FIG. 147

If the member BC in Fig. 147 (*a*) were cut, the truss would collapse. If—in place of the member BC—two equal and opposite forces, the magnitude of each of which is equal to the force exerted by the member BC, were introduced at points B and C, the truss would retain its original position. If the same procedure is applied to the members CF and EF, the truss will be divided into two parts, each of which will retain its original position. This construction is shown in Fig. 147 (*b*). In Fig. 147 (*d*) the right half of the truss is shown as a free body acted upon by two known forces and three unknown forces. The left half of the truss is shown as a free body in Fig. 147 (*c*).

The student should study the free-body diagrams of Figs. 147 (*c*) and 147 (*d*) very carefully, noting that the parts of the truss which have not been cut are performing their functions just as before. Therefore, in Fig. 147 (*d*) we can think of the free body as a rigid triangular block acted upon by two known forces and

three unknown forces. The free body of Fig. 147 (*c*) may be regarded as a rigid parallelogram acted upon by two known forces and three unknown forces. The unknowns will now be found by solving the free body of Fig. 147 (*d*).

The two unknowns *BC* and *CF* intersect at *C*. If an axis through *C* is selected as the axis of moments, these two unknowns will be eliminated from the moment equation.

$$\Sigma M_C = 0$$
$$5{,}000 \times 15 - EF \times 15 \times 0.866 = 0$$
$$EF = 5{,}775 \text{ lb, T.}$$

If an axis through point *F* is selected as the axis of moments, the forces *CF* and *EF* will be eliminated from the moment equation.

$$\Sigma M_F = 0$$
$$5{,}000 \times 22.5 - 6{,}000 \times 7.5 - BC \times 15 \times 0.866 = 0$$
$$BC = 5{,}195 \text{ lb, C.}$$

Since members *BC* and *EF* are horizontal, they have no vertical components; therefore, the vertical component of *CF* must balance the two known vertical forces.

$$\Sigma F_v = 0$$
$$CF \times 0.866 + 5{,}000 - 6{,}000 = 0$$
$$CF = 1{,}155 \text{ lb, C.}$$

EXAMPLE 2

Solve for the stresses in the members *a*, *b*, and *c* of the truss shown in Fig. 148 (*a*) by the method of sections.

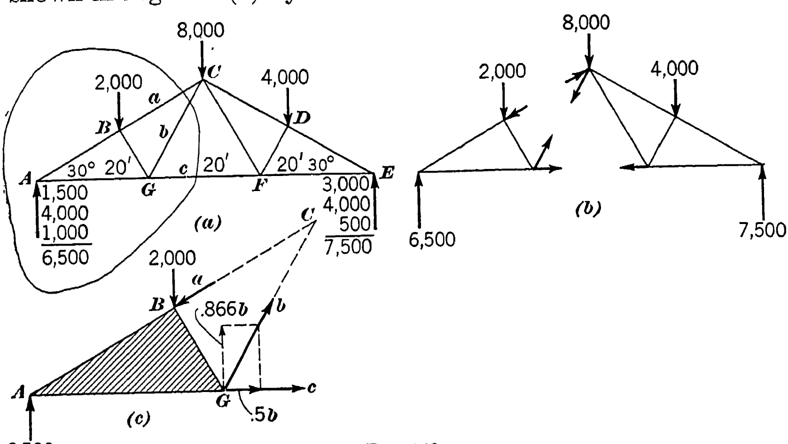


FIG. 148

The reactions may be obtained by moment equations or can be determined by inspection, if the principle of inverse proportion is employed.

After determining the reactions, cut the members a, b, and c, as indicated in Figs. 148 (a) and 148 (b). The part of the truss to the left of the cut members constitutes the free body which will be solved. The part to the right might be used for the free body, but it would involve one more known force; hence, the left half will be used.

In Fig. 148 (c) the free body for the left half of the truss can be considered a rigid triangular block acted upon by two known forces and three unknown forces. Since the unknown forces b and c intersect at point G, an axis through this point will be the first axis of moments.

$$\Sigma M_G = 0$$
$$a \times 10 - 6{,}500 \times 20 + 2{,}000 \times 5 = 0$$
$$a = 12{,}000 \text{ lb, C.}$$

Since the unknowns a and b intersect at C, an axis through that point will be selected.

$$\Sigma M_C = 0$$
$$c \times 20 \times 0.866 - 6{,}500 \times 30 + 2{,}000 \times 15 = 0$$
$$c = 9{,}535 \text{ lb, T.}$$

The unknowns a and c intersect at A. The unknown stress b may be resolved into a horizontal component and a vertical component, which act at the point G. This is indicated by the small parallelogram constructed at G in Fig. 148 (c). The horizontal component passes through A and therefore produces no moment with respect to an axis through the point A.

$$\Sigma M_A = 0$$
$$b \times 0.866 \times 20 - 2{,}000 \times 15 = 0$$
$$b = 1{,}730 \text{ lb, T.}$$

The stress b could also be determined by a vertical summation, as the vertical component of b must balance the two known vertical forces and the vertical component of a. This method involves the use of a computed stress, which is not good procedure if it can be avoided.

PROBLEMS

140. Solve the free body of Example 1 by using one moment equation and two summations.

141. Solve the free body of Example 2 by one moment equation and two summation equations.

142. Explain which solution is preferable, that given in the Examples or the methods used in Problems 140 and 141.

143. Compute the stresses in members *BC*, *BG*, *GH*, and *DF*, Fig. 149.

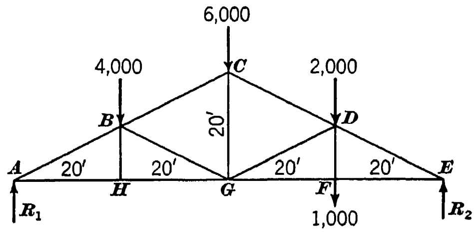


Fig. 149

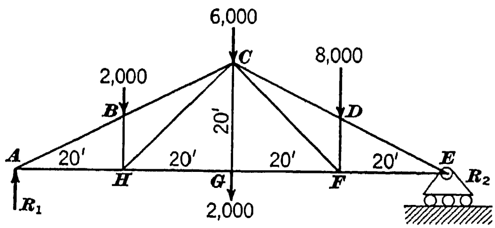


Fig. 150

144. Calculate the stresses in members *CD*, *CF*, *FG*, and *CG*, Fig. 150.

145. Compute the resultant reactions at *A* and *I* and also the stresses in members *CD*, *DG*, *FG*, and *BH*, Fig. 151.

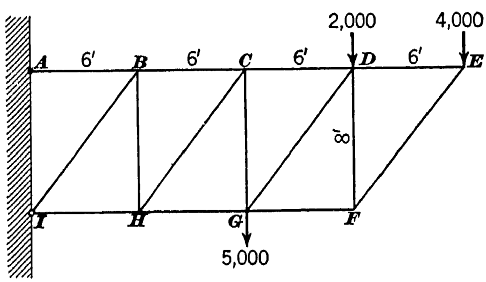


Fig. 151

47. **Redundancy.**—Some structures have members or supports which are unnecessary for the maintenance of equilibrium of the

structure. A common example of redundancy is the small flat-top highway bridge, like that shown in Fig. 152, which has two diagonals in each of its center panels; each diagonal is designed to carry tension only. One or the other of the diagonals in a panel is unnecessary at all times and is considered as not working since it buckles as soon as it is subjected to any compressive stress.

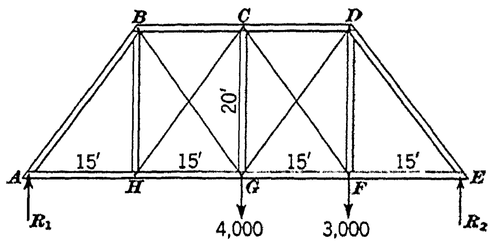


FIG. 152

If the assumption is made that only one diagonal in a panel acts for any given loading of the truss, the truss becomes statically determinate, and it can be solved by the ordinary methods of statics already developed.

There are certain other examples of redundancy, such as continuous trusses with several supports, and also various other built-up structures, such as the frame shown in Fig. 153, which do not yield to the approximate method just stated. The stresses set up in structures of this type must satisfy the conditions of equilibrium and also must be consistent with the deformations of the individual members of the structure, or obey what is known as the law of consistent deformation. The solution of such statically indeterminate structures will be left to more advanced books.

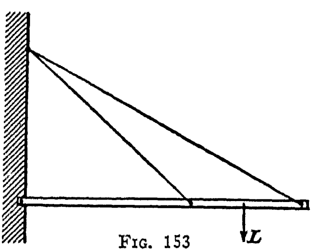


FIG. 153

PROBLEM

146. Determine the stresses in the two tension diagonals in Fig. 152.

48. **Multi-Force Members.**—Cranes, A frames, bents, and certain other built-up structures have their loads applied in a different manner from that which is used in the case of ordinary roof and bridge trusses. As explained in Art. 34, the roof and

bridge trusses are built up of triangular units, with the loads applied at the corners or joint pins. All the members of a truss are either simple tension members or simple compression members, which are considered to extend only from pin to pin.

In cranes and certain other structures this type of construction is not possible. Some of the members of the structure must resist bending and shear. The bending and shear are caused by the application of the loads or forces at points between the ends of the members.

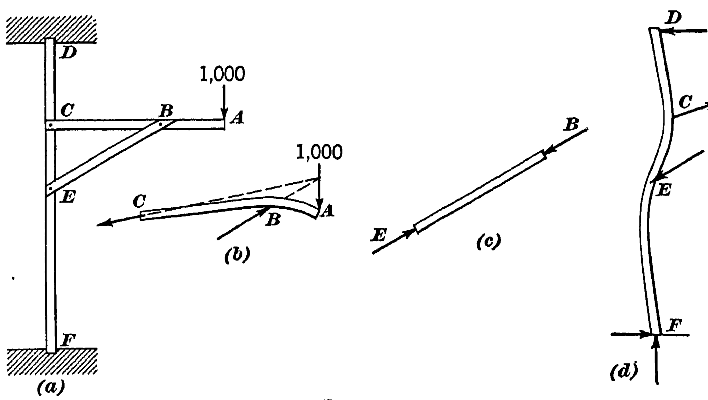


FIG. 154

In a frame the weight of each member is generally small enough to be neglected or divided between the pins at the ends of the member. Where the weight of a member of a frame is large, there will generally be sufficient bending to necessitate consideration of the piece as a multi-force member.

If a member is subjected to bending, a section cannot be taken through the member, because the stress in the member cannot be represented by a single force acting along the line connecting the two pins, as was possible in the case of the two-force members of a roof or bridge truss.

Any member which has forces[2] *acting on it at points other than the pins at the two ends of the member is known as a multi-force member.*

Consider the simple crane of Fig. 154 (*a*) with bearings at D and F. It is evident that the member ABC must be continuous if it is to support the 1,000-lb load. Because of the action of the

[2] One of which does not act along the axis of the member.

forces at the points A, B, and C, the member AC will tend to take the shape shown in Fig. 154 (b). It will be readily observed that the resultant effect of a member of this type cannot be represented by a single force. The member is therefore a multi-force member.

The member BE has forces applied at B and E only. This member is in direct compression. It is a two-force member.

The post DF must also be a continuous member, or the crane will not stand up. This post has forces acting at D, C, E, and F which cause the member to bend. It will tend to assume a shape approximately as shown in Fig. 154 (d).

The method of procedure for solving cranes, A frames, and other structures which involve multi-force members is as follows:

(1) Consider the entire structure as a free body, and solve for the external reactions as completely as possible.

(2) Take out as a free body a multi-force member which has known forces acting upon it. Solve for as many of the unknown forces as possible.

(3) Take out a second multi-force member as a free body, and solve it as completely as possible. Continue this procedure until all desired information is obtained.

EXAMPLE 1

Solve for the external reactions and the pin pressures at points B, C, and F in the crane shown in Fig. 155 (a).

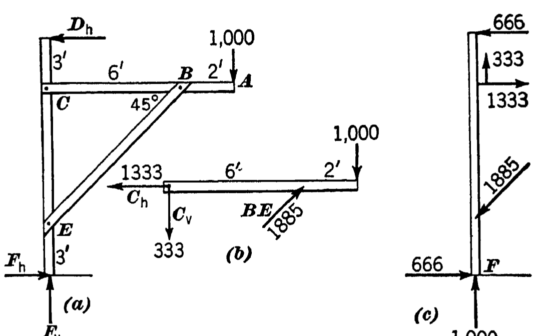
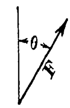


FIG. 155

Consider the entire structure as a free body, and take moments with respect to an axis through F.

$$\Sigma M_F = 0$$
$$12D_h - 1{,}000 \times 8 = 0$$
$$D_h = 666 \text{ lb} \leftarrow$$
$$\Sigma F_h = 0$$
$$F_h = 666 \text{ lb} \rightarrow$$
$$\Sigma F_v = 0$$
$$F_v = 1{,}000 \text{ lb} \uparrow$$

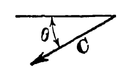
$$F = \sqrt{1{,}000^2 + 666^2}$$
$$= 1{,}200 \text{ lb}$$
$$\tan\theta = \frac{666}{1{,}000}$$
$$\theta = 33.67°$$



Take out the member ABC as a free body, Fig. 155 (b). The pin pressure at C is represented by its horizontal and vertical components C_h and C_v.

$$\Sigma M_C = 0$$
$$0.707\ BE \times 6 - 1{,}000 \times 8 = 0$$
$$BE = 1{,}885 \text{ lb, C.}$$
$$\Sigma F_h = 0$$
$$1{,}885 \times 0.707 - C_h = 0$$
$$C_h = 1{,}333 \text{ lb} \leftarrow$$
$$\Sigma F_v = 0$$
$$1{,}885 \times 0.707 - C_v - 1{,}000 = 0$$
$$C_v = 333 \text{ lb} \downarrow$$

$$C = \sqrt{1{,}333^2 + 333^2}$$
$$= 1{,}373 \text{ lb}$$
$$\tan\theta = \frac{333}{1{,}333} = 0.25$$
$$\theta = 14.03°$$



In Fig. 155 (c) the post is shown as a free body with all forces acting. It will be noticed that the directions of the forces acting at point C and E are opposite from those in Fig. 155 (b). These directions follow from the law of action and reaction. For every action there must be an equal and opposite reaction.

EXAMPLE 2

Solve for the reactions at A and C, and also for the amount and direction of the pin pressure at B, in Fig. 156 (a).

This structure consists of two multi-force members, since each of the members is subjected to bending. The reactions at A and C must have both horizontal and vertical components if the structure is to stand up.

When we attempt to apply rule 1, we find that, if the entire structure is used as the free body, there will be four unknown forces. Since only three independent equations can be written, this free body cannot be solved.

$$\Sigma F_h = 0 \text{ gives } A_h = C_h$$
$$\Sigma F_v = 0 \text{ gives } A_v + C_v - 300 = 0$$

Apply rule 2 and take out each multi-force member as a free body. Fig. 156 (*b*) shows the member AB as a free body held in equilibrium by one known force and four unknown forces. Fig. 156 (*c*) shows the member BC as a free body also held in equilibrium by one known force and four unknown forces. It will be observed, however, that the forces at B in Fig. 156 (*c*) are equal and opposite to those at B in Fig. 156 (*b*), because they are actions and reactions.

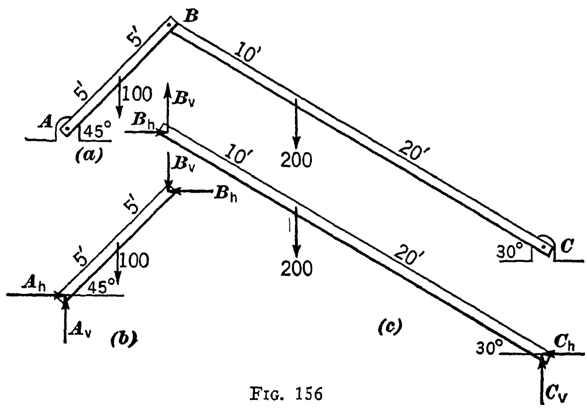


FIG. 156

The directions of the horizontal components at B are evident from inspection, but the directions of the vertical components are not so easily seen. Assume that in Fig. 156 (*b*) the component B_v acts down; then in Fig. 156 (*c*) the component B_v must act up. With Fig. 156 (*b*) as the free body, sum the moments with respect to an axis through A.

$$\Sigma M_A = 0$$

$$7.07\ B_h - 7.07\ B_v - 100 \times 3.53 = 0 \qquad (1)$$

With Fig. 156 (*c*) as the free body,

$$\Sigma M_C = 0$$

$$-15\ B_h - 25.98\ B_v + 200 \times 17.3 = 0 \qquad (2)$$

Divide equation (1) by 7.07 and equation (2) by 15.

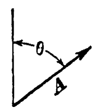
$$\begin{array}{r} B_h - \ B_v - 49.8 = 0 \\ \underline{-B_h - 1.73\ B_v + 231 = 0} \\ -2.73\ B_v + 181.2 = 0 \\ B_v = 66 \text{ lb} \downarrow \end{array}$$

The positive sign of B_v indicates that the correct assumption as to direction of B_v was made in the original equations. In Fig. 156 (*b*),

$$\Sigma F_v = 0$$
$$A_v - 100 - 66 = 0$$
$$A_v = 166 \text{ lb } \uparrow$$

$$\Sigma M_A = 0$$
$$7.07\, B_h - 7.07 \times 66 - 100 \times 3.53 = 0$$
$$B_h = 116 \text{ lb } \leftarrow$$

$$\Sigma F_h = 0$$
$$A_h = 116 \text{ lb } \rightarrow$$
$$A = \sqrt{116^2 + 166^2} = 202 \text{ lb}$$
$$\tan\theta = \frac{116}{166} = 0.698$$
$$\theta = 34.9°$$

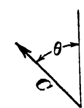


In Fig. 156 (*c*),

$$\Sigma F_v = 0$$
$$C_v + 66 - 200 = 0$$
$$C_v = 134 \text{ lb } \uparrow$$

$$\Sigma F_h = 0$$
$$B_h = C_h = 116 \text{ lb } \leftarrow$$
$$C = \sqrt{116^2 + 134^2} = 177.5 \text{ lb}$$

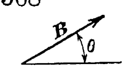
$$\tan\theta = \frac{116}{134} = 0.866$$
$$\theta = 40.9°$$



The resultant pin pressure at B is

$$B = \sqrt{66^2 + 116^2} = 133.5 \text{ lb}$$

$$\tan\theta = \frac{66}{116} = 0.568$$
$$\theta = 29.6°$$



PROBLEMS

147. Compute the reactions at B and E, and also the amounts of the pin pressures at C and D, for the crane in Fig. 157. *Ans.* $B_H = 618$ *lb;* $E_H = 618$ *lb;* $E_V = 1{,}000$ *lb;* $AC = 2{,}000$ *lb;* $AD = 1{,}732$ *lb.*

148. Compute the stress in DE, the pin pressure at C, and the reactions at B and F in Fig. 158.

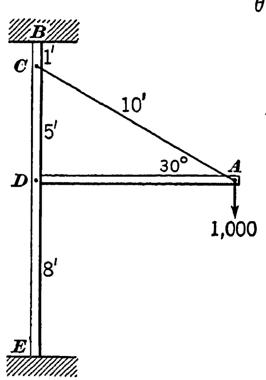


FIG. 157

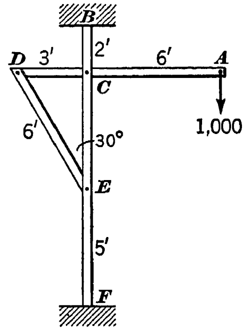


FIG. 158

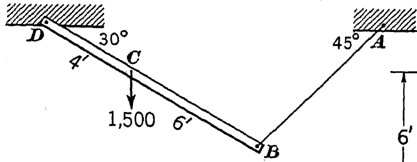


FIG. 159

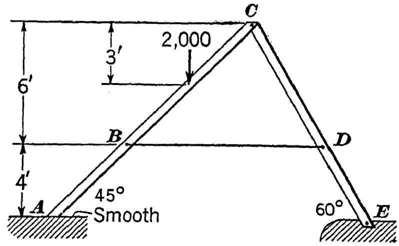


FIG. 160

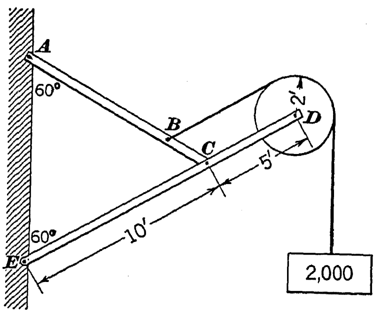


FIG. 161

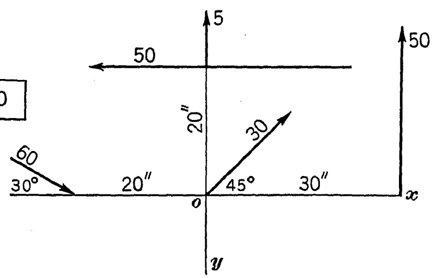


FIG. 162

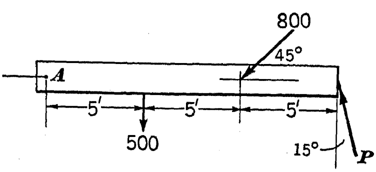


FIG. 163

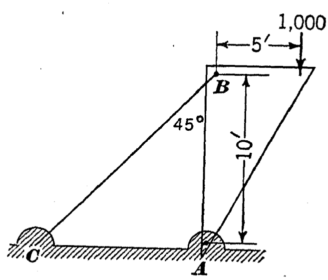


FIG. 164

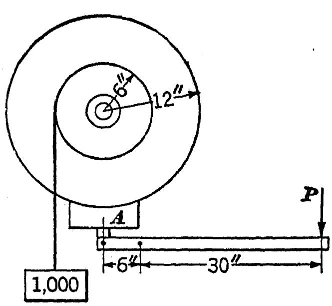


FIG. 165

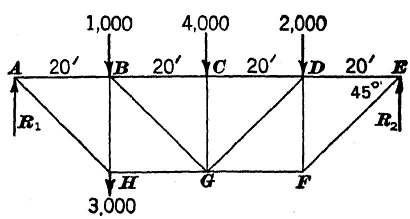


FIG. 166

149. Compute the tension in the cord AB and the pin pressure at D in Fig. 159.

150. The A frame in Fig. 160 rests on a smooth floor at A. Determine the stress in BD and the pin pressure at C.

151. Compute the components of the reactions at A and E, Fig. 161.

REVIEW PROBLEMS

152. Compute the amount and position of the resultant of the force system shown in Fig. 162. *Ans. 517 lb; 60 in.; 63.33°.*

153. Compute the amount of the force P in Fig. 163 if the bar is in equilibrium. When P is acting, what are the amount and direction of the reaction at A?

154. Determine the tension in the cord BC and the vertical and horizontal components of the reaction at A in Fig. 164.

155. If, in Fig. 165, the frictional force developed at A is 0.4 times the normal pressure, what force will be required at P to just prevent motion? What will the horizontal and vertical components of the bearing reaction be?

156. Solve for the stresses in all members of the truss shown in Fig. 166. *Ans. AB = 5,500 lb, C.; AH = 7,780 lb, T.; BH = 2,500 lb, C.; BC = 7,000 lb, C.; BG = 2,125 lb, T.; GH = 5,500 lb, T.; CD = 7,000 lb, C.; CG = 4,000 lb, C.; DG = 3,540 lb, T.; DE = 4,500 lb, C.; DF = 4,500 lb, C.; EF = 6,375 lb, T.; FG = 4,500 lb, T.*

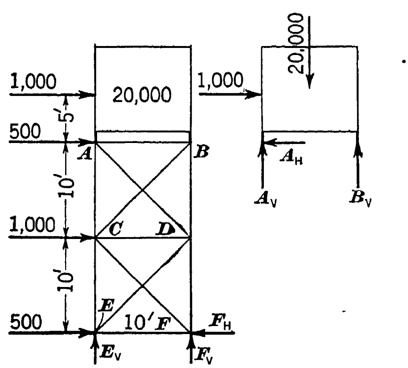


FIG. 167

157. In the water-tank frame of Fig. 167 assume that the diagonals can take tension only. Solve for the reactions and all stresses.

158. Solve for the stresses in all members of the truss shown in Fig. 104.

159. Determine the stresses in all members of the truss shown in Fig. 106. *Ans. AB = 15,000 lb, T.; BC = 13,000 lb, T.; CD = 6,000 lb, T.; DE = 4,000 lb, T.; EF = 3,464 lb, C.; FG = 9,820 lb, C.; GH = 4,545 lb, C.; CF = 9,235 lb, T.; CG = 9,805 lb, C.; CH = 3,464 lb, T.; BH = 3,464 lb, C.; DF = 3,464 lb, C.; IG = 17,600 lb, C.; AH = 1,085 lb, C.*

160. Solve for the stresses in members *en*, *np*, and *pj* in Fig. 103 by the method of sections.

161. Find the stresses in the diagonal members of Fig. 105 by the method of sections.

162. By the method of sections find the stresses in *cn*, *no*, and *ol* of Fig. 108.

163. Find the stresses in *ft*, *ts*, and *sk* of the truss shown in Fig. 108. *Ans.* *ft* = 6,700 *lb*, *C.*; *st* = 2,600 *lb*, *T.*; *sk* = 4,500 *lb*, *T.*

164. Solve for the stresses in *BC*, *CG*, and *GF* in Fig. 107.

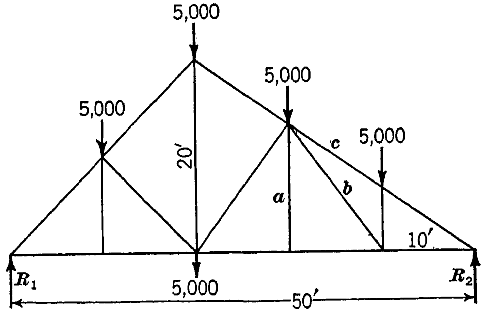


FIG. 168

165. Solve for the stresses in members *a*, *b*, and *c* in Fig. 168.

166. In Fig. 169, *EF* can take compression only. Solve for the reactions at *A* and *F*, and also the stresses in *AB* and *BE*.

167. Solve for the stresses in *a*, *b*, and *c*, Fig. 170. *Ans.* *a* = 8,900 *lb*, *C.*; *b* = 7,300 *lb*, *C.*; *c* = 0.

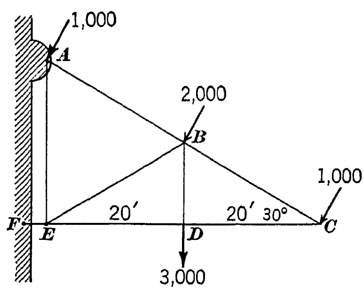


FIG. 169

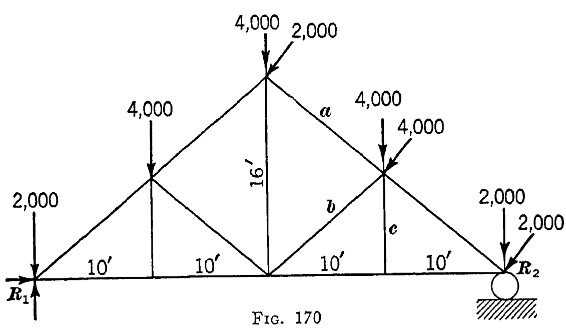


FIG. 170

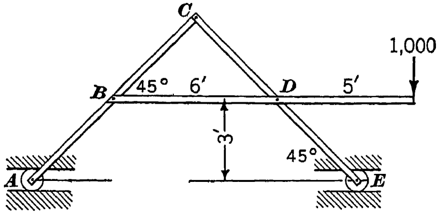


FIG. 171

168. Determine the horizontal and vertical components of the pin pressures at all pins in the A frame of Fig. 171.

169. Determine the components of the pin pressures at A, B, and C in the two-member arch of Fig. 172.

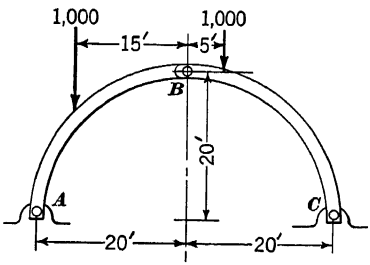


FIG. 172

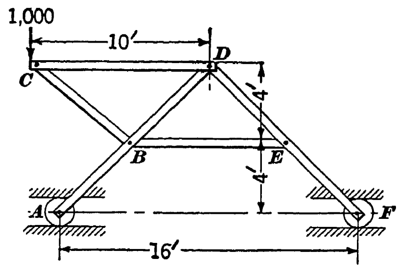


FIG. 173

170. Solve for the stresses in BC, CD, and BE in Fig. 173.

171. Determine all forces acting on each member of Fig. 174. *Ans.* $A_V = 100$ *lb;* $BE = 325$ *lb, T.;* $C_V = 400$ *lb;* $C_H = 325$ *lb;* $F_V = 400$ *lb.*

172. Solve for all components acting at all pins in Fig. 175.

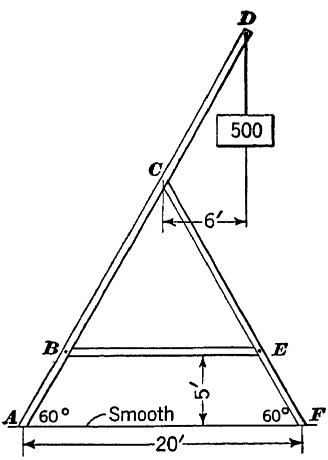


FIG. 174

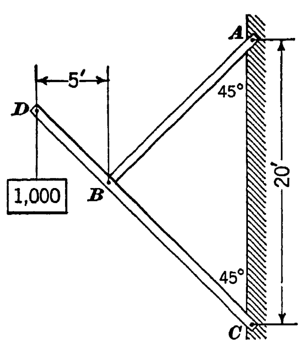


FIG. 175

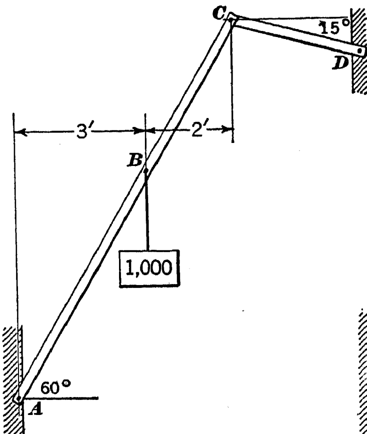


FIG. 176

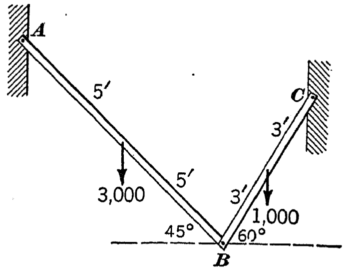


FIG. 177

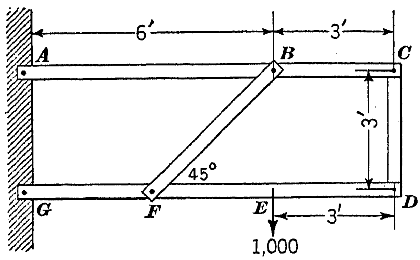


FIG. 178

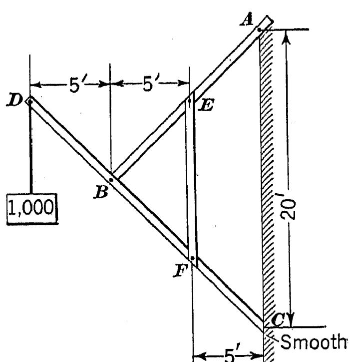


FIG. 179

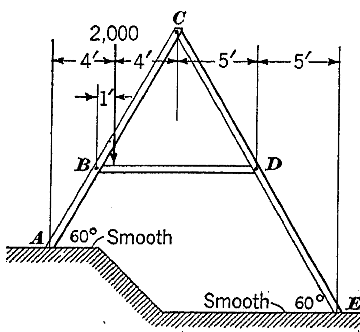


FIG. 180

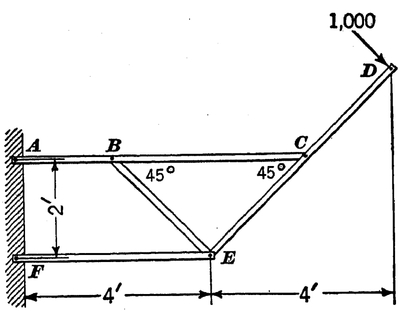


FIG. 181

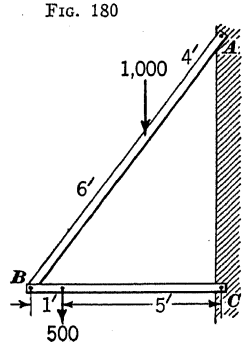


FIG. 182

173. Determine the stress in *CD* and the components of the reaction at *A* in Fig. 176.

174. Solve for the components of the forces acting at *A*, *B*, and *C* in Fig. 177.

175. Compute the stresses in members *CD* and *BF* and also find the components of the reactions at *A* and *G* in Fig. 178. *Ans.* $A_H = 2{,}000$ *lb;* $A_V = 667$ *lb;* $G_H = 2{,}000$ *lb;* $G_V = 1{,}667$ *lb;* $BF = 2{,}830$ *lb, C.;* $CD = 1{,}333$ *lb, T.*

176. Solve for all components of all forces acting on the members of Fig. 179. How does the addition of member *EF* change the structure from that shown in Fig. 175? What effect does *EF* have on the reactions at *A* and *C*?

177. The A frame shown in Fig. 180 rests on smooth planes at *A* and *E*. Determine the components of all forces acting on the members.

178. Solve for all forces acting on the left post of the bent shown in Fig. 119 (*a*). Using the entire bent as a free body, solve for R_{1x} and R_{1y}, the components at the base of the post. Next take out the post as a free body, as indicated in Fig. 120 (*a*).

179. Solve for all forces acting on the left post and the stress in the main horizontal member *d* of the bent shown in Fig. 121. *Ans.* $a = 1{,}000$ *lb, T.;* $b = 13{,}065$ *lb, C.;* $c = 278$ *lb, T.;* $d = 6{,}750$ *lb, T.*

180. Determine the components of the reactions at *A* and *F* and also the stress in *BE* in Fig. 181.

181. Compute the components of the reactions at *A* and *C* in Fig. 182.

182. Solve for the components of the forces acting on the pins of the three-hinged arch shown in Fig. 183.

183. Solve for the stresses in *BC* and *CF* in Fig. 184. *Ans.* $BC = 10{,}845$ *lb, C.;* $CF = 12{,}335$ *lb, T.*

184. Determine the stresses in members *CD, DE, CE, BC*, and *CF* of the mine head frame shown in Fig. 185.

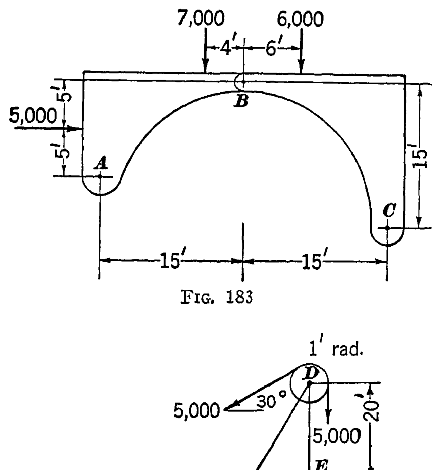


FIG. 183

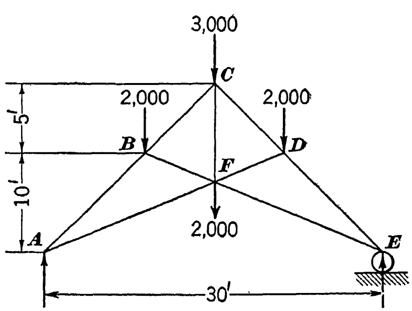


FIG. 184

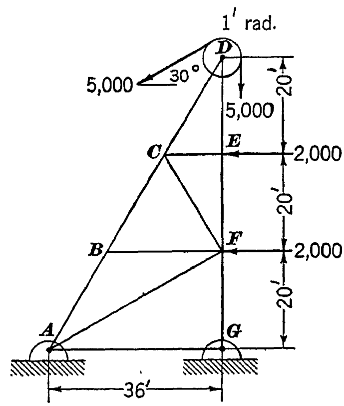


FIG. 185

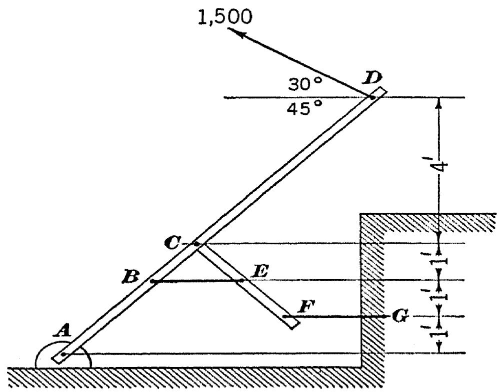


FIG. 186

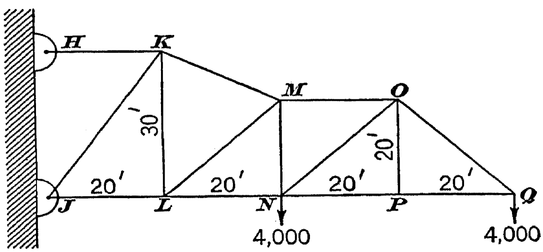


FIG. 187

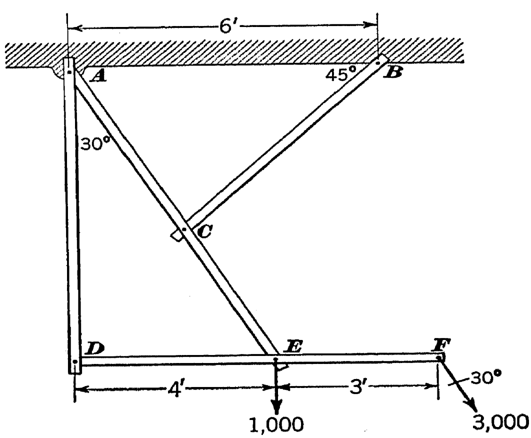


FIG. 188

185. Solve for the stresses in *FG* and *BE*, Fig. 186; also find the horizontal and vertical components of the reaction at *A*.

186. Determine the stresses in members *MO*, *MN*, *LN*, and *LM* of the truss shown in Fig. 187.

187. Solve for the horizontal and vertical components of the pin reaction at *A* and for the stresses in members *AD* and *BC* in Fig. 188.

188. Compute the stresses in members *AC*, *BC*, *CF*, and *DF* in Fig. 189.

189. Solve for the stresses in *BC* and *CD* and the reactions at *A* and *E* in Fig. 190.

190. Determine the horizontal and vertical components of the forces on all the pins in Fig. 191.

191. Solve for the components of the pin reactions at *A* and *G* and the stresses in members *BF* and *CE* in Fig. 192.

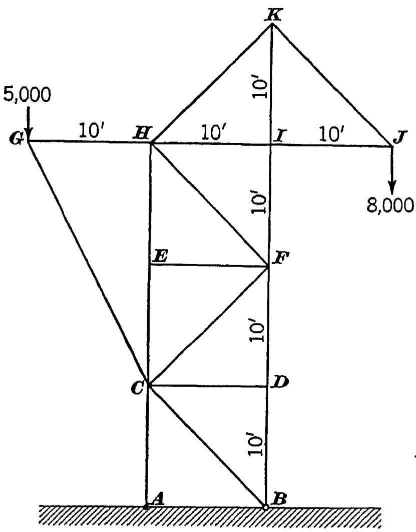


FIG. 189

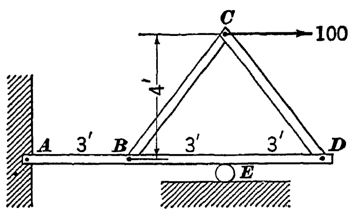


FIG. 190

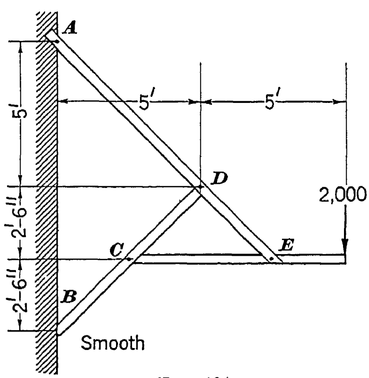


FIG. 191

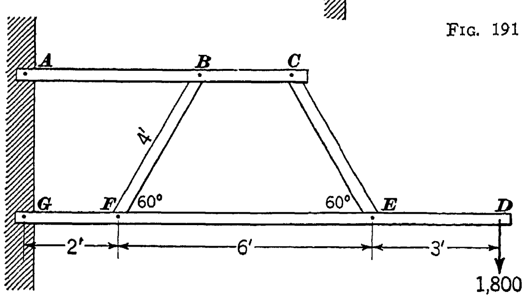


FIG. 192

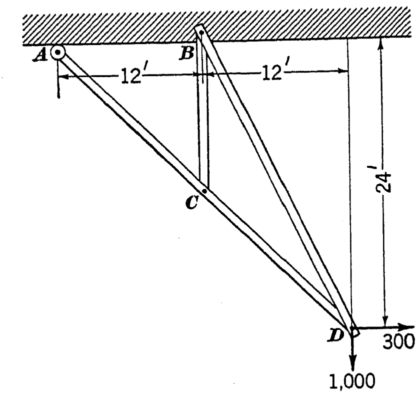


FIG. 193

192. Determine the force at A and the stresses in BC and BD in Fig. 193.

193. Compute the resultant reactions at pins A and G and also the stresses in CF, CH, and BH in Fig. 194.

194. Compute the stresses in members BC, CG, and GF in Fig. 195.

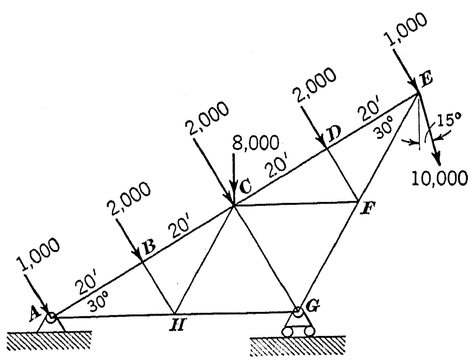


FIG. 194

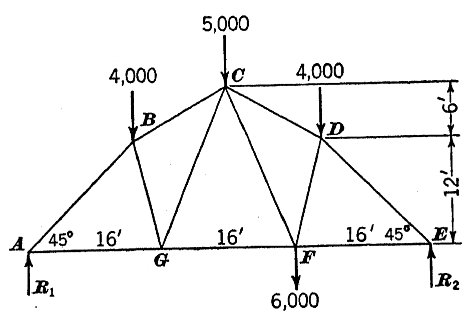


FIG. 195

195. Solve for the components of the reaction at F and the stresses in CH, GH, and CD in Fig. 196.

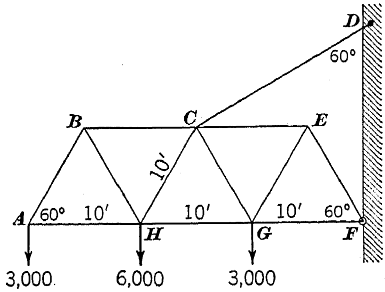


FIG. 196

196. Determine the horizontal and vertical components of the pin reactions at A, C, and E of the three-hinged arch in Fig. 197 and also the stresses in members BG and CF.

197. Determine the stresses in members AB, BC, CF, and GI in Fig. 198.

198. Compute the components of the pin reactions at pins A and E and also the stresses in members BE and BD in Fig. 199.

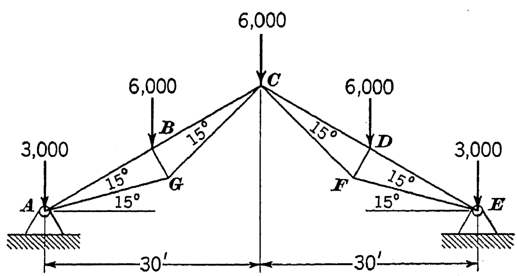


FIG. 197

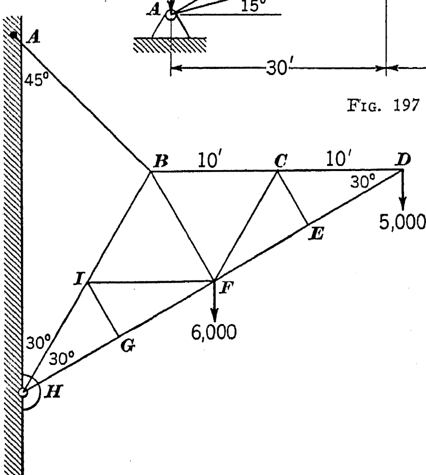


FIG. 198

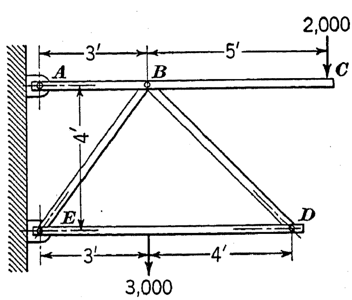


FIG. 199

CHAPTER 6

NON-COPLANAR FORCE SYSTEMS BY GRAPHICAL METHODS

49. Resultant and Equilibrium of Non-Coplanar, Parallel Force Systems.—The solution of a system of parallel forces in space does not require the development of any new methods. The student is referred at this time to the methods developed in Chapter 3, Arts. 22 and 23, for coplanar force systems.

The resultant of any system of non-coplanar, parallel forces is a single force or a couple. If the resultant is a single force, its magnitude is simply the algebraic sum of the parallel forces. If the resultant is a couple, the algebraic sum of the forces is zero.

If the system of forces is projected onto each of two planes that are parallel to the given forces, such as the X–Y plane and the Y–Z plane, where the Y axis is parallel to the forces, each projection will be a coplanar, parallel system which can be solved by the method of Art. 22 or Art. 23. The solution of each projection will give the distance to the resultant of the system from the Y axis. Therefore, the two solutions definitely locate the line of action of the resultant of the system in space.

If the resultant of the given parallel system is a couple, the solution of each of the two projections will be a couple. The two couples may be combined into a resultant couple by the method suggested in Art. 58.

For equilibrium of a non-coplanar, parallel system, there are three conditions to be satisfied: $\Sigma F=0; \Sigma M_x=0; \Sigma M_z=0$. Therefore, there cannot be more than three unknown forces if the system is to be solved.

The method of solution is as follows: Project the system onto a plane which is parallel to the forces. This plane should be chosen so that the projections of two of the unknowns coincide. The third unknown can be determined from this projection by the method developed in Art. 23. A second projection of the system may now be made. This projection will contain only two unknown forces. These unknowns can be determined by the same method as was used for the first unknown.

50. Resolution of a Force Into Three Component Forces. Let vector F in Fig. 200 represent any force in space. This force may be resolved into components parallel to any three lines in the following manner. Let OX, OY, and OZ represent the lines to which the components are to be parallel. Through F pass a plane perpendicular to the plane XOZ. The force F is thus resolved into two components F_{xz} and F_y. The component F_{xz} is now resolved by the parallelogram construction into F_x and F_z. The results are: $F_x = F \cos \alpha$; $F_y = F \cos \beta$; $F_z = F \cos \gamma$.

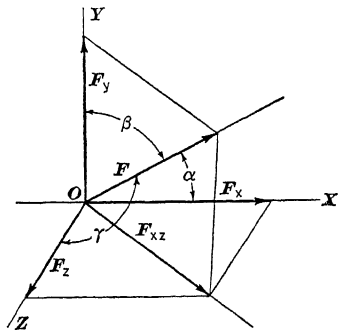


FIG. 200

51. Resultant of Non-Coplanar, Concurrent Force Systems. To find the resultant of a non-coplanar, concurrent force system, we might construct a force polygon in space, in a manner similar to the construction described in Art. 12 for concurrent, coplanar force systems. The closing line of this polygon in space would be the resultant of the system in amount and direction. Since the resultant and all component forces must pass through the point of concurrence, a line drawn through this point parallel to the closing line of the force polygon in space will be the line of action of the resultant force.

The above method, in the general case, is apt to involve rather difficult construction work. A better solution is to resolve each force into its X, Y, and Z components by the method of Art. 50. All the X components may then be combined into a single component ΣF_x; all the Y components into a component ΣF_y; and the Z components into a component ΣF_z. In this manner the system is reduced to three forces. Any two of these forces may be combined by the parallelogram method into a resultant, and this resultant may be combined with the third component force to determine the resultant of the entire system. This resultant will pass through the point of concurrency.

52. Equilibrium of a Non-Coplanar, Concurrent Force System.—If a system of non-coplanar, concurrent forces is in equilibrium, the resultant of the system must be zero; that is, $R=0$.

Therefore, $\Sigma F_x=0$; $\Sigma F_y=0$; $\Sigma F_z=0$. Thus, there can be no resultant force acting along any one of three intersecting lines, one of which does not lie in the plane of the other two. This implies three independent conditions of equilibrium or a possibility of solving for three unknown quantities, such as the amounts of three forces or the amount of one force and the amount and direction of a second.

If $R=0$ the force polygon in space must be a closed figure. The projection of this space polygon on any plane will be a closed figure. Since this projected polygon is a coplanar diagram, the unknown quantities in such a diagram cannot exceed two, if it is to close. If the plane of projection is selected so that the projections of two of the unknown forces coincide, the projected polygon will contain only two unknown quantities and can be made to close, and these unknowns will then be determined.

EXAMPLE 1

Determine the compression in the legs of the tripod shown in Fig. 201 (a).

Project all forces on the plane through ABF. The free body for the projected force system is shown in Fig. 201 (b). In this projection the forces AC and AD coincide. Force F of Fig.

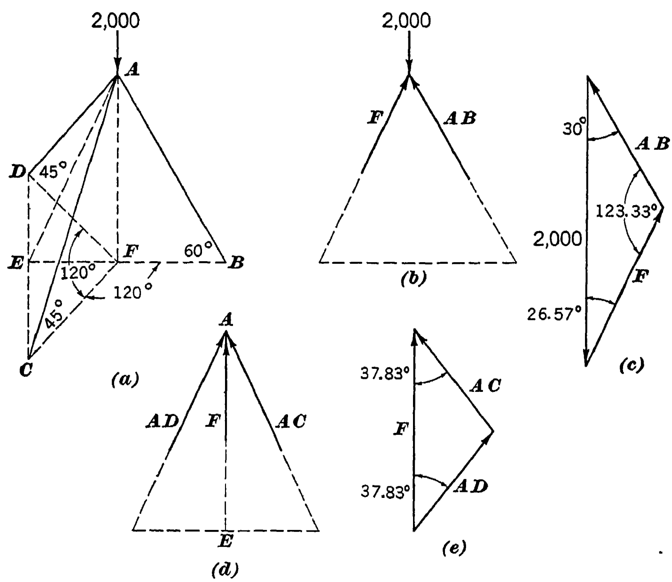


FIG. 201

201 (*b*) is the resultant of forces *AC* and *AD*; or forces *AC* and *AD* may be considered as being replaced by the single force *F*, which acts along the line *AE*. In Fig. 201 (*c*) is shown the force triangle for the free body of Fig. 201 (*b*). The values of *AB* and *F* are given by this triangle. For the determination of *AC* and *AD* the plane *ACD* is shown in true size in Fig. 201 (*d*), with forces *F*, *AC*, and *AD* acting at *A*. In the force triangle of Fig. 201 (*e*) the vector *F* is laid off to any convenient scale and parallel to *F* of Fig. 201 (*d*). Vectors *AC* and *AD* are drawn through the ends of *F* and parallel to *AC* and *AD* of Fig. 201 (*d*). The vectors *AC* and *AD* represent the forces *AC* and *AD* to the scale which was used for *F*. Thus, force *F* has been resolved into components along *AC* and *AD*. The required results are: *AB* = 1,075 lb, C.; *AC* = *AD* = 758 lb, C.

EXAMPLE 2

Determine the stresses in members *AB*, *AC*, and *AD* of the shear-legs crane shown in Fig. 202 (*a*).

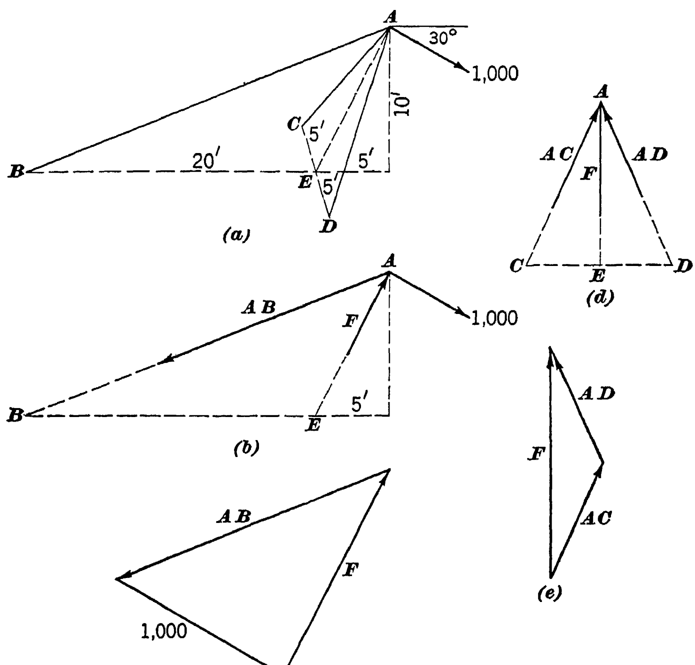


Fig. 202

Project all forces on the plane ABE. The free body shown in Fig. 202 (b) is obtained from this projection. Force F represents the resultant of forces AC and AD and acts along the line AE. Fig. 202 (c) is the force triangle for the three forces acting at A in Fig. 202 (b). Fig. 202 (d) shows the plane ACD in true size. In Fig. 202 (e) force F is laid down to scale, and vectors AC and AD are drawn through the ends of F parallel to AC and AD in Fig. 202 (d). The values of AC and AD are thus determined. The results are: $AB=1{,}500$ lb, T.; $AC=AD=650$ lb. C.

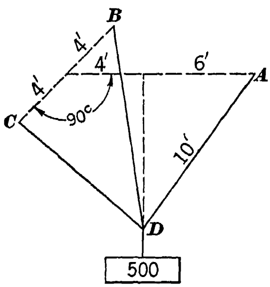


FIG. 203

PROBLEMS

199. A weight of 500 lb is supported by three ropes attached to a horizontal ceiling at points A, B, and C, as shown in Fig. 203. Determine the tension in each of the ropes. *Ans.* $AD=250$ *lb;* $BD=184$ *lb;* $CD=184$ *lb.*

200. Determine the stresses in the members AB, AC, and AD of the wall-frame shown in Fig. 204.

201. If the line of action of the 1,000-lb force shown in Fig. 205 is horizontal and 30° back of the plane ACE, what stress will each guy wire carry?

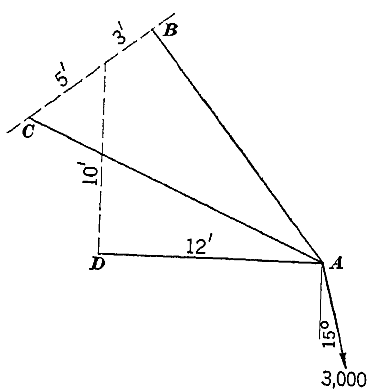


FIG. 204

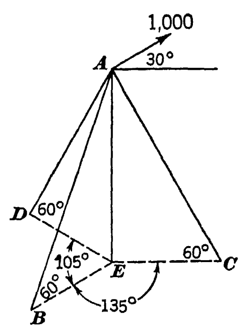


FIG. 205

53. Resultant and Equilibrium of Non-Coplanar, Non-Concurrent Force Systems.—The resultant of this most general form of the force system may be a single force or a couple, but it is most often expressed as a resultant force acting through a selected point O and a resultant couple. A set of X, Y, and Z axes is

drawn with the selected point O as the intersection of the axes. Each of the given forces is then resolved into components parallel to the X, Y, and Z axes, as described in Art. 50. Each of these components may be further resolved into a parallel force of the same magnitude passing through the selected point O and a couple whose magnitude and sense are equal to the magnitude and sense of the moment of the original component force about an axis that passes through O and is perpendicular to the plane of the original component (see Art. 29). The resultant force for the system is then the vector sum of all the forces acting through the point O, as stated in Art. 51; and the resultant couple is the vector sum of all the couples (see Art. 58). A graphical solution of such a problem generally is far too involved to have much practical value.

In Art. 52 the condition necessary for equilibrium of a non-coplanar, concurrent force system was that the force polygon in space close. This condition implied that $R=0$, or that $\Sigma F_x=0$, $\Sigma F_y=0$, and $\Sigma F_z=0$; therefore, there can be no resultant force in any direction.

For equilibrium of a non-coplanar, non-concurrent system the above condition must be satisfied; that is, the force polygon in space must close. In addition there must be no tendency to rotate about any axis; or $\Sigma M_x=0$, $\Sigma M_y=0$, and $\Sigma M_z=0$. This means that the funicular polygon in space must close.

The usual method of procedure is to project the force system onto each of the coordinate planes in turn. Each of these projections will be a coplanar force system in equilibrium and may be solved as such. No projection can contain more than three unknown quantities; otherwise, it will be impossible to solve that projection. (See Arts. 17 and 33 on coplanar force systems.)

There are three independent conditions of equilibrium for each projection:

$$XY \text{ Plane, } \Sigma F_x=0;\ \Sigma F_y=0;\ \Sigma M_z=0.$$
$$YZ \text{ Plane, } \Sigma F_y=0;\ \Sigma F_z=0;\ \Sigma M_x=0.$$
$$XZ \text{ Plane, } \Sigma F_x=0;\ \Sigma F_z=0;\ \Sigma M_y=0.$$

There are three equations in the above group which are duplicated. Therefore, there are only six independent equations; or a non-coplanar, non-concurrent force system cannot have more than six unknown quantities, if it is to be solved.

EXAMPLE 1

Solve for the stresses in all members of the crane shown in Fig. 206 (a) if the plane ABE bisects angle CED.

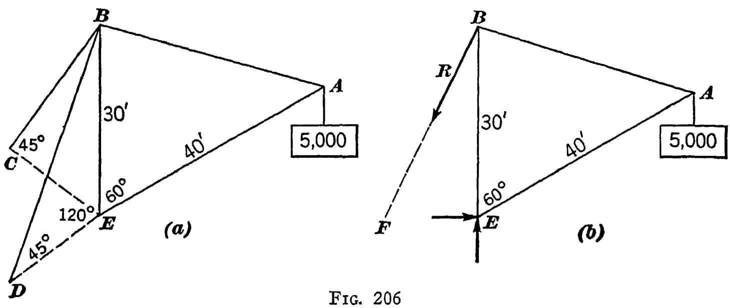


FIG. 206

Project the force system on the plane through ABE. The members BC and BD are replaced by their resultant R acting along AF in Fig. 206 (b). The force systems at A and B in Fig. 206 (b) can now be solved by the methods for coplanar, concurrent systems. These solutions are given in Fig. 207 (a) and 207 (b).

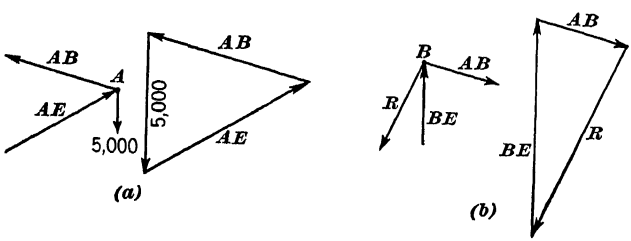


FIG. 207

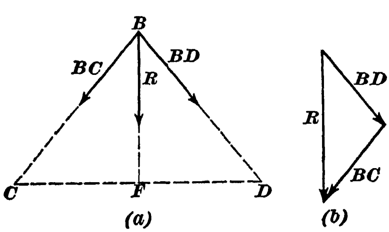


FIG. 208

In Fig. 208 (a) the plane BCD of Fig. 206 (a) is shown in its true size; and in this projection R, which is the resultant of BC and BD, is shown. Fig. 208 (b) shows the solution for the stresses

in members BC and BD. The results are: $AB = 6{,}000$ lb, T.; $AE = 6{,}666$ lb, C.; $BE = 13{,}210$ lb, C.; $BC = 8{,}160$ lb, T.; $BD = 8{,}160$ lb, T.

EXAMPLE 2

Determine the tension T and the horizontal and vertical reactions at the bearings A and B of the jack-shaft shown in Fig. 209 (a).

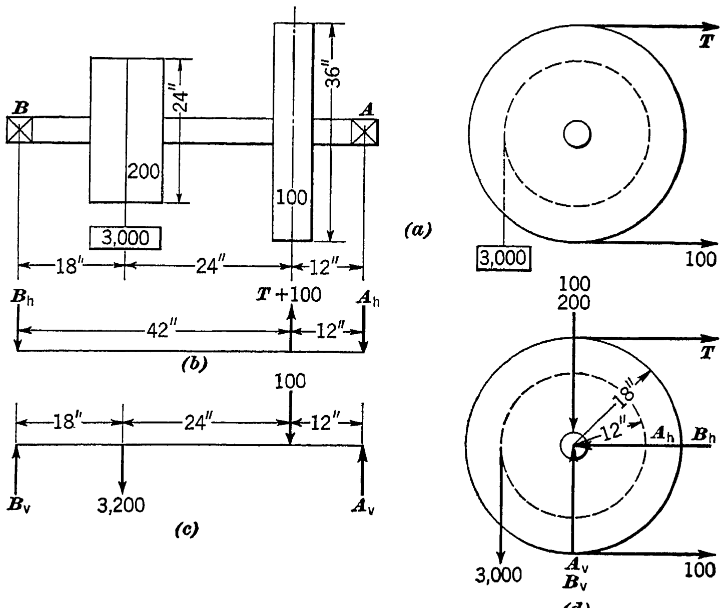


FIG. 209

In Fig. 209 (b), (c), and (d) are shown the projections of the shaft on the three coordinate planes. By taking moments with respect to the center of the shaft in Fig. 209 (d) the value of the belt pull T can be obtained.

$$\Sigma M_o = 0$$
$$18\,T - 3{,}000 \times 12 - 100 \times 18 = 0$$
$$T = 2{,}100 \text{ lb}$$

With T known, the reactions A_h and B_h can be determined by the method of Art. 25, as shown in Fig. 210 (a). The line AC is laid down to scale equal to 2,200 lb. The construction gives

$AB = 490$ lb, which is B_h, and $BC = 1{,}710$ lb, which is A_h. In the same manner, in Fig. 210 (*b*), the 3,200- and 100-lb loads of Fig. 209 (*c*) are resolved into the vertical components of the reactions at A and B. Vector DF represents 3,200 lb to any convenient scale. The length of DE, or 1,067 lb, is the portion of the 3,200 lb carried at A_v, and $EF = 2{,}133$ lb is the portion carried at B_v. In a like manner GI represents the 100-lb load to any convenient scale; GH is the portion carried at B_v, and HI is the portion acting at A_v. Thus, $A_v = DE + HI = 1{,}145$ lb, and $B_v = EF + GH = 2{,}155$ lb.

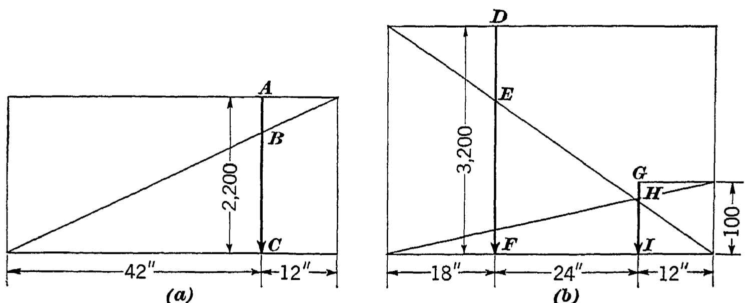


FIG. 210

PROBLEMS

202. Solve Example 1 if the boom is lowered to a horizontal position. Let BC and BD be at an angle of 30° with the horizontal and the angle between them 135° instead of 120°. *Ans.* $AB = 8{,}333$ *lb, T.*; $AE = 6{,}666$ *lb, C.*; $BC = 10{,}030$ *lb, T.*; $BD = 10{,}030$ *lb, T.*

203. Solve Example 2 if the pulley and the bearing A are interchanged, and also the belt pulls act up at 30° with the horizontal.

54. Determination of the Maximum Stresses in the Backstays of a Crane.

—The determination of the maximum stresses in the backstays of a crane can best be explained by study of the following example, careful inspection of the drawings shown in its graphical explanation, and solution of the problems offered.

EXAMPLE

For the crane shown in Fig. 211 (*a*), determine the maximum tensile stress which can be caused in member EG if the boom ABC, carrying the 5,000-lb load, is assumed free to swing through 360°. Member ABC weighs 1,000 lb and member BD weighs 500 lb.

Fig. 211 (b) shows the post and boom held in equilibrium by the reactions at F and the "overturning force" OTF at E. By inverse proportion, combine the 5,000-lb load with the 1,000 lb due to the weight of ABC, and obtain the 6,000-lb resultant R_1, acting as shown. This resultant R_1 is now combined with the 500-lb weight to get $R_2 = 6{,}500$ lb. The mast and the boom are now held in equilibrium by the known resultant R_2, the reaction at F, and the force OTF. Extend the force OTF and R_2 until they intersect at O. The reaction at F must pass through F and O; therefore, its line of action is determined. Starting at O, lay off a 6,500-lb vector, to any convenient scale, along the line of action of R_2. Through the end of the 6,500-lb vector draw a horizontal line to intersect the line of action of F. The force OTF is thus found to be 2,973 lb.

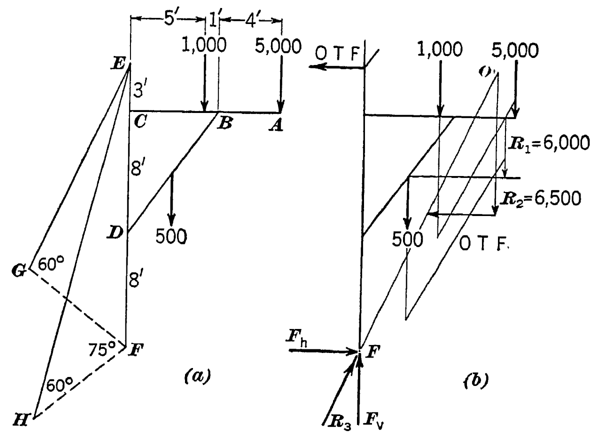


Fig. 211

Fig. 212 (a) shows the forces acting at E projected on a horizontal plane through E. The force OTF has been reversed and placed so that it is perpendicular to the plane of EH. This is the position of the boom which will cause the maximum tensile stress in EG. The force triangle for this system is shown in Fig. 212 (a). From this triangle, $EG_h = 3{,}075$ lb, T. and $EH_h = 797$ lb, C.

Study and experimentation with Fig. 212 (a) will indicate that EG_h has its maximum value when the boom is in the position shown, or at 90° with the plane of EH_h. It will be observed that, if the angle grows larger than 90°, EH_h will become smaller,

reaching zero magnitude when the boom (2,973-lb vector) is in line with EG_h. In Fig. 212 (*b*) are shown the two force triangles for obtaining the true values of the stresses in EG and EH. The results are: $EG = 6{,}150$ lb, T. and $EH = 1{,}594$ lb, C.

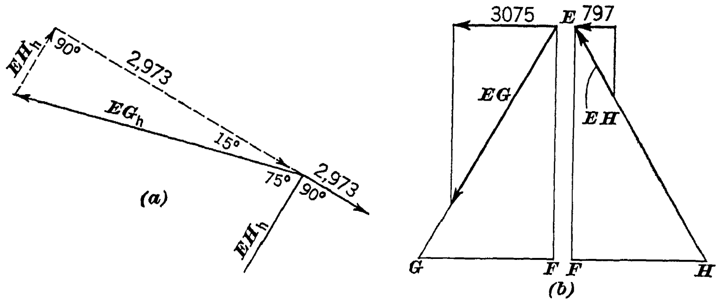


FIG. 212

It will be noted that in Fig. 212 (*a*) the vector EG_h acts away from E and vector EH_h acts toward E. Hence, the stress in EG is tension and that in EH is compression.

PROBLEMS

204. In the example just solved, change the load to 10,000 lb and the angle between the backstays to 120°. Solve for the maximum tension in EH. What position of the boom will cause maximum compression in EH? *Ans.* $EH = 12{,}950$ *lb, T.;* $EG = 6{,}475$ *lb, T.*

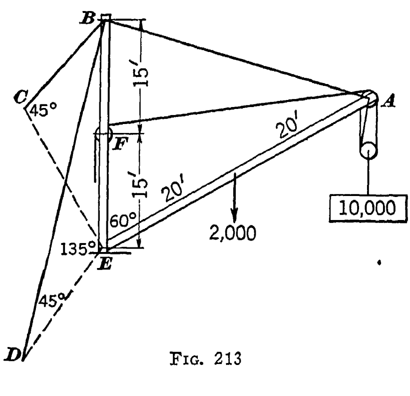


FIG. 213

205. In Fig. 213 the boom weighs 2,000 lb. The rope is fastened at A and passes twice around the lower pulley, over pulleys A and F, and then down to a hoisting engine which is not shown. All pulleys are 2 ft in diameter. Place the boom in the position which will cause the maximum stress in BC. With the boom in this position, determine the stresses in AB, BC, and BD.

REVIEW PROBLEMS

206. Determine graphically the location of the resultant of the force system shown in Fig. 214. *Ans.* $x = \frac{1}{3}$ *ft;* $z = 3\frac{1}{3}$ *ft.*

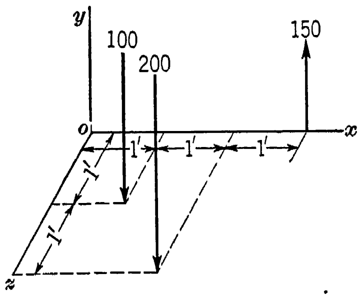


FIG. 214

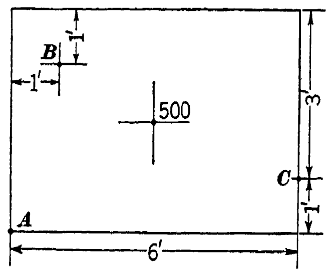


FIG. 215

207. Fig. 215 represents a rectangular table top with a load of 500 lb placed at the center. The table is supported at the points A, B, and C. Determine by graphical construction the amount of each reaction.

208. Determine graphically the amounts of the stresses in AB, AC, and AD, Fig. 216.

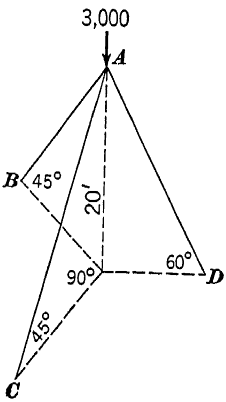


FIG. 216

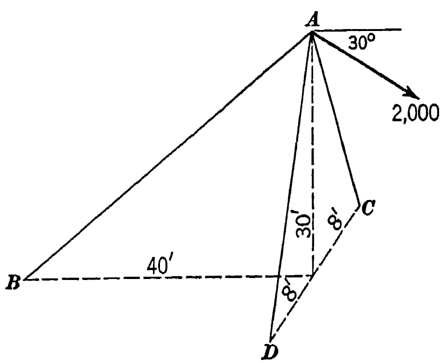


FIG. 217

209. By graphical construction solve for the stresses in AB, AC, and AD in Fig. 217. The 2,000-lb force acts in the vertical plane through AB, at an angle of 30° with the horizontal.

210. Solve Problem 209 if the 2,000-lb force is turned 15° toward D.

CHAPTER 7

NON-COPLANAR FORCE SYSTEMS BY MATHEMATICAL METHODS

55. Resolution of a Force Into Three Components.—Any force in space can be broken up into any desired number of components; however, the components usually desired are those parallel to the three coordinate axes. The method of resolution is clearly shown in Fig. 200, Art. 50. The component parallel to the X axis is given by the equation $F_x = F \cos \alpha$, where α is the angle between the force F and the X axis. In a similar manner $F_y = F \cos \beta$ and $F_z = F \cos \gamma$.

56. Moment of a Force With Respect to Any Line in Space. Let F, Fig. 218, be the given force, and let OX, OY, and OZ be any rectangular axes drawn through any point O. The moment of the force F with respect to each of the three axes can then be easily found. Resolve the force F into components parallel to each of the three axes, and obtain F_x, F_y, and F_z. The three components of F may act at any point A along the line of action of the resultant force F. The perpendicular distance to A from each of the three axes is shown. The moment of the force F with respect to each axis is then equal to the moment of its components with respect to that axis. These moments are given by the following equations:

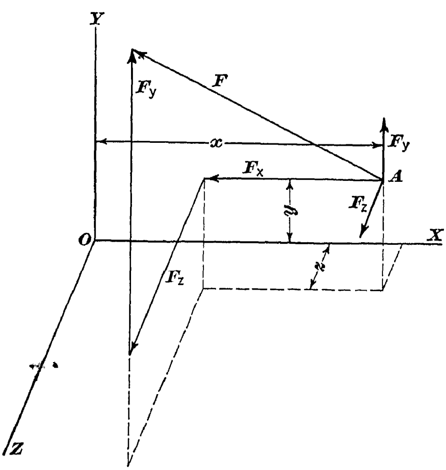


FIG. 218

$$M_x = F_y z - F_z y$$
$$M_y = F_x z + F_z x$$
$$M_z = -F_x y - F_y x$$

57. The Principle of Moments.—Art. 16 demonstrates Varignon's Theorem for coplanar forces. This theorem states that *the moment of a resultant force with respect to any axis perpendicular to the plane of the resultant force is equal to the algebraic sum of the moments of the component forces with respect to the same axis.*

This theorem can be extended to the general case. *For any force system in space, the moment of the resultant with respect to any axis in space is equal to the algebraic sum of the moments of the component forces with respect to the same axis.*

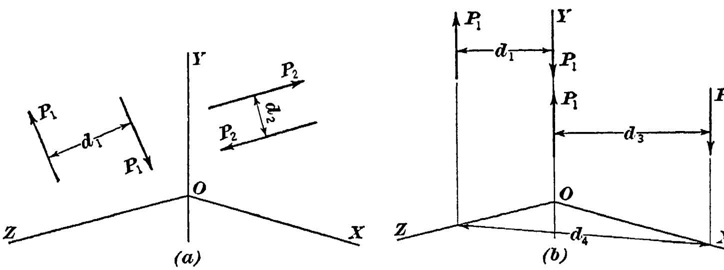


FIG. 219

58. Resultant of Couples in Space.—Couples in a plane were discussed in Arts. 28 and 29. In Fig. 219 (*a*), P_1d_1 and P_2d_2 are any two couples in planes which intersect along the line OY. In Art. 28 it was shown that any couple can be moved about in its plane or into a parallel plane at will. It was shown also that changing the magnitude of the two forces and the perpendicular distance between the forces does not change the effect of the couple if the product of either force of a couple and the perpendicular distance between the two forces remains a constant quantity.

In Fig. 219 (*b*) both couples in (*a*) have been moved about in their planes and the couple P_2d_2 has been adjusted so that $P_1d_3 = P_2d_2$. The equal and opposite forces P_1 acting along OY, Fig. 219 (*b*), cancel each other, the other two forces P_1 being left to form the new resultant couple P_1d_4.

Since any couple can be moved about in its plane or into parallel planes without changing its effect, it is possible to combine any number of couples into a single resultant couple. Each of the given couples can be moved about so that all the vectors representing the individual couples pass through a common point. These vectors can then be combined by the method given in Art.

61 for concurrent forces in space. The resultant vector will then represent the resultant couple.

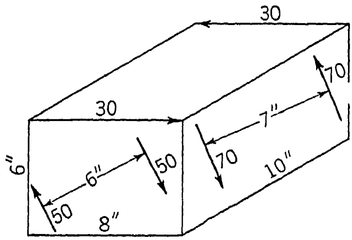


FIG. 220

PROBLEM

211. Compute the magnitude of the resultant couple in Fig. 220, and determine whether the direction of its moment is clockwise or counter-clockwise.

59. **Resultant of Parallel Forces in Space.**—The magnitude of the resultant of a system of parallel forces in space is given by the algebraic sum of the component forces. The coordinates of the resultant on a plane may be determined by applying the principle of moments: *The moment of the resultant with respect to any line is equal to the algebraic sum of the moments of the component forces with respect to the same line.* By the application of this principle the distances to the resultant from any two lines in space can be found, and the resultant will be definitely located. The coordinates axes which are perpendicular to the lines of action of the forces are the lines of reference usually selected.

EXAMPLE

Compute the amount and location of the resultant of the force system shown in Fig. 221.

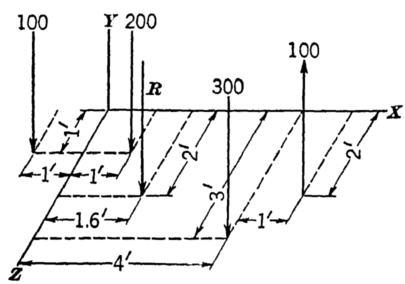


FIG. 221

$$R = 500 \text{ lb} \downarrow$$

$$\Sigma M_z = Rx$$

$$x\,500 = 200\times1 - 100\times1 + 300\times4 - 100\times5$$

$$x = 1.6 \text{ ft}$$

$$\Sigma M_x = Rz$$

$$z\,500 = 100\times1 + 200\times1 + 300\times3 - 100\times2$$

$$z = 2 \text{ ft}$$

PROBLEM

212. Determine the amount and position of the resultant of the force system shown in Fig. 222. *Ans.* $x=2.54$ *ft*; $z=0.076$ *ft.*

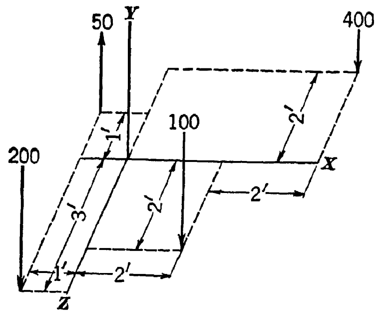


FIG. 222

60. **Equilibrium of Parallel Force Systems in Space.**—If a parallel force system in space is in equilibrium, the resultant force must be zero; or $R=0$. In addition, there must be no tendency to rotate about either of any two intersecting axes which lie in a plane perpendicular to the lines of action of the forces. If the forces are parallel to the Y axis, the conditions for equilibrium are $\Sigma F_y=0$; $\Sigma M_x=0$; $\Sigma M_z=0$. If $\Sigma F_y=0$, but either ΣM_x or ΣM_z is not zero, the system is not in equilibrium but is equivalent to a couple.

Since there are three equations to be satisfied for equilibrium, three unknown quantities can be determined.

EXAMPLE 1

Fig. 223 (a) represents a horizontal table top which is supported at the points A, B, and C. Determine the reactions at these points.

Assume the X and Y axes as shown in Fig. 223 (a). Using the projection shown in Fig. 223 (b) and a line through A as the axis of moments, we obtain the following equation:

$$B+4C-500\times2=0 \quad (1)$$

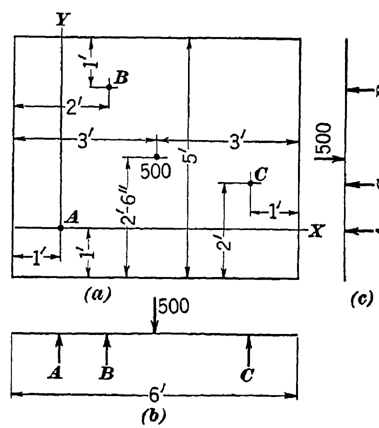


FIG. 223

Using the projection shown in Fig. 223 (c), we may write a second equation with a line through A as the axis of moments.

$$3B+C-500\times1.5=0 \tag{2}$$

Solve equations (1) and (2) for the values of the reactions at B and C. It is found that $B=181.9$ lb and $C=204.5$ lb.

$$\Sigma F_y=0$$
$$A+181.9+204.5-500=0$$
$$A=113.6 \text{ lb}$$

The student will note that the important point in the solution just given is the proper selection of axes. The moment axes should be so selected that they will intersect at the point of application of one of the unknown forces, in order to eliminate this unknown from both moment equations.

Since there are only three conditions of equilibrium for a non-coplanar, parallel force system, more than three supports produce a redundant condition, and the problem becomes indeterminate. Certain reasonable assumptions in regard to the distribution of the loading can sometimes be made, in order to reduce the number of unknowns to not more than three.

PROBLEMS

213. A triangular table top has loads at D and E, as shown in Fig. 224. What are the reactions at the points A, B, and C? *Ans.* $A=613$ *lb;* $B=320$ *lb;* $C=567$ *lb.*

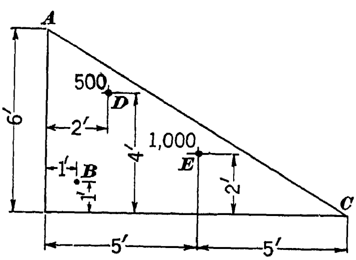


Fig. 224

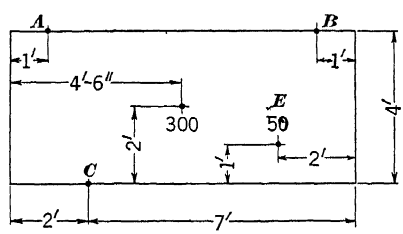


Fig. 225

214. A horizontal trap-door has hinges at A and B, as shown in Fig. 225. The door weighs 300 lb, and an additional weight of 50 lb is placed at E. What vertical force must be supplied at C to support the door in a horizontal position? What forces are acting at the hinges?

215. Solve Problem 214 if the door is turned about edge AB at an angle of 15° below the horizontal, and the force at C is normal to the door.

216. A circular table top 6 ft in diameter has a load of 400 lb at the center. The three legs are on a circle 5 ft in diameter, and they are spaced 90°, 130°, and 140° apart. Determine the amount of each reaction.

217. Locate a 600-lb load on the table in Problem 216 so that all three legs carry the same load.

61. Resultant of Concurrent Forces in Space.—Since all the forces of a concurrent system must pass through a common point, no rotation can be produced. The resultant of such a system of forces is a single force passing through the point of concurrence.

There are several ways of determining the resultant force. The best method is to resolve each force of the system into its X, Y, and Z components at the point of concurrence. The resultant of the X components, or ΣF_x, is found by adding algebraically all the X components. In a similar manner, ΣF_y and ΣF_z are found. The resultant of the system is given by the relation

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

The direction of the resultant force R with respect to each of the coordinate axes is given by $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, where α, β, and γ are the angles between R and the X, Y, and Z axes.

$$\cos \alpha = \frac{\Sigma F_x}{R}; \ \cos \beta = \frac{\Sigma F_y}{R}; \ \cos \gamma = \frac{\Sigma F_z}{R}$$

In the example and problems which follow, all forces are concurrent at the origin O and pass through the points which are designated by their X, Y, and Z coordinates.

EXAMPLE

For the force system shown in Fig. 226, determine the amount and the direction of the resultant. The forces are: 60 lb (*4, 6, 2*); 40 lb (*3, 4, 5*); 50 lb (*5, 9, 6*).

$$\sqrt{4^2+2^2+6^2} = 7.48 \text{ ft}$$

$$\sqrt{3^2+5^2+4^2} = 7.07 \text{ ft}$$

$$\sqrt{5^2+6^2+9^2} = 11.9 \text{ ft}$$

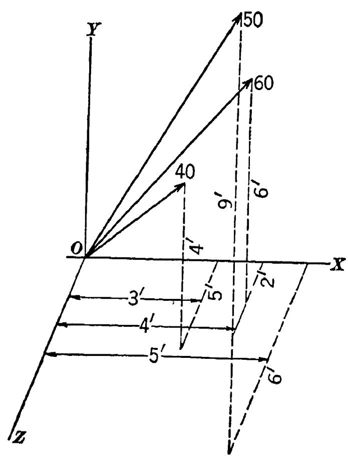


FIG. 226

FORCE	COMPONENTS x	y	z
60	$\frac{60\times4}{7.48}=32$	$\frac{60\times6}{7.48}=48.2$	$\frac{60\times2}{7.48}=16$
40	$\frac{40\times3}{7.07}=17$	$\frac{40\times4}{7.07}=22.6$	$\frac{40\times5}{7.07}=28.3$
50	$\frac{50\times5}{11.9}=20.9$	$\frac{50\times9}{11.9}=37.7$	$\frac{50\times6}{11.9}=25.2$
	$\Sigma F_x=\overline{70}$	$\Sigma F_y=\overline{108.5}$	$\Sigma F_z=\overline{69.5}$

$$R=\sqrt{70^2+108.5^2+69.5^2}=146.5\text{ lb}$$

$$\cos\alpha=\frac{70}{146.5}=0.477;\ \cos\beta=\frac{108.4}{146.5}=0.74;\ \cos\gamma=\frac{69.5}{146.5}=0.475$$

$$\alpha=61.5^\circ\qquad\beta=42.3^\circ\qquad\gamma=61.6^\circ$$

PROBLEMS

218. Determine the value of the resultant, and its angle with each of the coordinate axes, if the following forces act through the origin. 100 lb (*6, 8, 10*); 60 lb (*5, 4, 3*); 80 lb (*8, 5, −3*). *Ans. 211 lb; 44.9°; 51.6°; 70.1°.*

219. Solve for the amount and direction of the resultant of the following forces: 70 lb (*7, 8, 9*); 80 lb (*−8, 4, −3*); 100 lb (*6, −4, −4*).

62. Equilibrium of Concurrent Forces in Space.—If the resultant of a non-coplanar, concurrent force system is zero, or $R=0$, the system is in equilibrium. Then $\Sigma F_x=0$, $\Sigma F_y=0$, and $\Sigma F_z=0$; and there are three independent equations for equilibrium which may be solved for three unknown quantities. Since the system is in equilibrium, it also follows from the principle of moments (Art. 57) that the algebraic sum of the moments of all forces of the system with respect to any axis in space is equal to zero.

It is often possible to project a non-coplanar, concurrent force system onto a plane in such a manner that the projections of two unknown forces coincide. The projection then obtained is a coplanar, concurrent force system with two unknowns. This projection may be solved by any of the methods used in Chapter 2.

EXAMPLE 1

Solve for the stresses in the members AB, AC, and AD of the frame shown in Fig. 227.

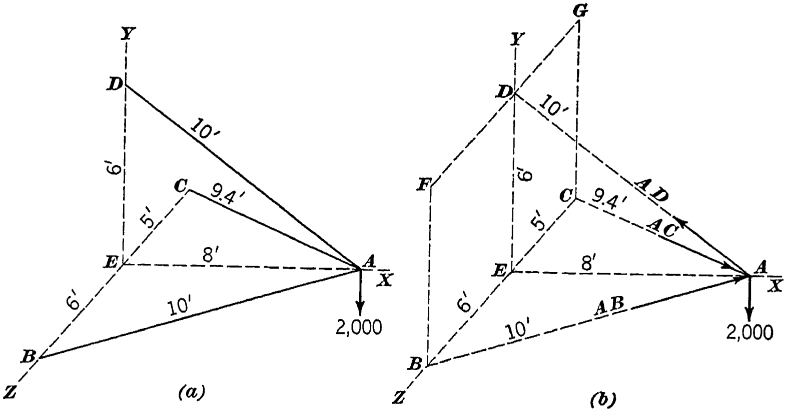


FIG. 227

FIRST METHOD

In Fig. 227 (*b*),

$$\Sigma F_y = 0$$
$$0.6\ AD - 2{,}000 = 0$$
$$AD = 3{,}333 \text{ lb, T.}$$

$$\Sigma F_x = 0$$
$$0.8\ AB + \frac{8}{9.4}\ AC - 3{,}333 \times 0.8 = 0$$

$$\Sigma F_z = 0$$
$$0.6\ AB - \frac{5}{9.4}\ AC = 0$$
$$AB = 1{,}515 \text{ lb, C.}$$
$$AC = 1{,}710 \text{ lb, C.}$$

SECOND METHOD

Project the force system onto the vertical plane ADE and the horizontal plane ABC. This produces Fig. 228 (*a*) and Fig. 228 (*b*).

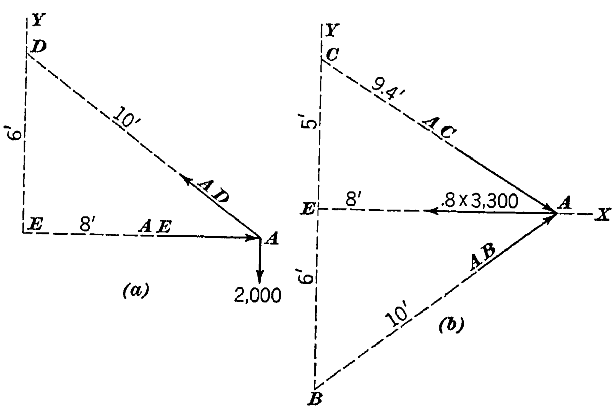


FIG. 228

In Fig. 228 (*a*),

$$\Sigma F_y = 0$$
$$0.6\ AD - 2{,}000 = 0$$
$$AD = 3{,}333 \text{ lb, T.}$$

In Fig. 228 (*b*),

$$\Sigma F_y = 0$$
$$0.6\ AB - \frac{5}{9.4}\ AC = 0$$

$$\Sigma F_x = 0$$
$$0.8\ AB + \frac{8}{9.4}\ AC - 0.8 \times 3{,}333 = 0$$
$$AB = 1{,}515 \text{ lb, C. and } AC = 1{,}710 \text{ lb, C.}$$

It will be observed that these two methods give identical equations.

THIRD METHOD

In Fig. 227 (*b*) use the line BC as a moment axis.

$$\Sigma M_{BC} = 0$$
$$0.8\ AD \times 6 - 2{,}000 \times 8 = 0$$
$$AD = 3{,}333 \text{ lb, T.}$$

$$\Sigma M_{BF} = 0$$
$$\frac{8}{9.4}\ AC \times 11 - 0.8 \times 3{,}333 \times 6 = 0$$
$$AC = 1{,}710 \text{ lb, C.}$$

$$\Sigma M_{CG} = 0$$
$$0.8\ AB \times 11 - 0.8 \times 3{,}333 \times 5 = 0$$
$$AB = 1{,}515 \text{ lb, C.}$$

EXAMPLE 2

Determine the stress in each of the members AB, AC, and AD, Fig. 229 (*a*).

This problem may be solved by any one of the methods used in Example 1. A method slightly different from those of the previous examples will now be illustrated.

Project the force system onto a plane passing through ABE, Fig. 229 (*a*). This will produce the projection shown in Fig. 229 (*b*). The unknowns AC and AD are now represented by AE.

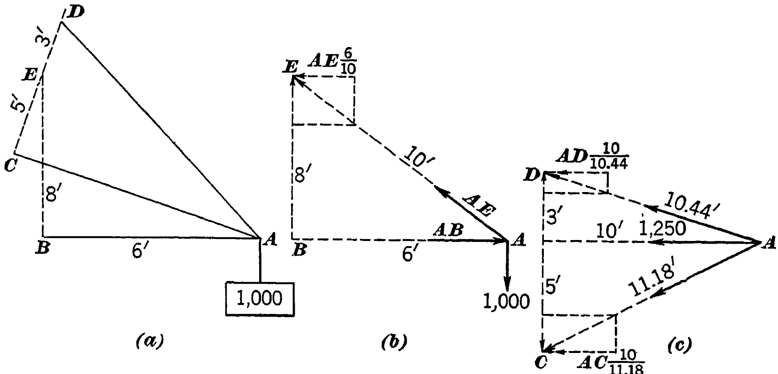


FIG. 229

$$\Sigma M_E = 0$$
$$8\ AB - 1{,}000 \times 6 = 0$$
$$AB = 750 \text{ lb, C.}$$

$$\Sigma M_B = 0$$
$$AE \times \frac{6}{10} \times 8 - 1{,}000 \times 6 = 0$$
$$AE = 1{,}250 \text{ lb, T.}$$

The plane ACD of Fig. 229 (a) is shown in its true size in Fig. 229 (c), with the resultant of AC and AD, or 1,250 lb, T., acting along AE. *This projection is not in equilibrium;* but, by the principle of moments, the moment of the 1,250-lb force is equal to the sum of the moments of the two component forces with respect to any axis normal to the plane ABC.

$$\Sigma M_C = 0$$
$$\frac{10}{10.44}\ AD \times 8 = 1{,}250 \times 5$$
$$AD = 818 \text{ lb, T.}$$

$$\Sigma M_D = 0$$
$$\frac{10}{11.18}\ AC \times 8 = 1{,}250 \times 3$$
$$AC = 525 \text{ lb, T.}$$

PROBLEMS

220. In Fig. 227 (a), change the distance CE to 6. Determine the stresses in AB, AC, and AD. *Ans.* $AB = 1{,}667$ *lb, C.;* $AC = 1{,}667$ *lb, C.;* $AD = 3{,}333$ *lb, T.*

221. Solve for the stresses in AB, AC, and AD in the shear-legs frame shown in Fig. 230.

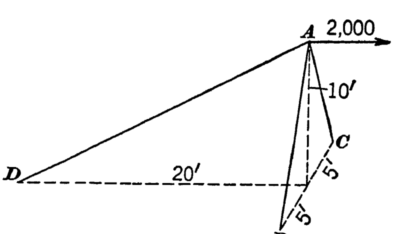


FIG. 230

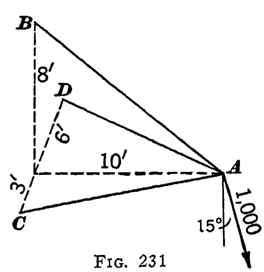


FIG. 231

222. Solve for the stresses in all members of the structure shown in Fig. 231.

223. In Fig. 232 a vertical pole is shown supported by three equally spaced guy wires AB, AC, and AD, each making an angle of 60° with the ground and each capable of carrying tensile stress only. The 2,000-lb pull acts at A 15° below the horizontal and 30° toward B from the plane AED. Determine the tension in each guy wire, and also the compression in the post.

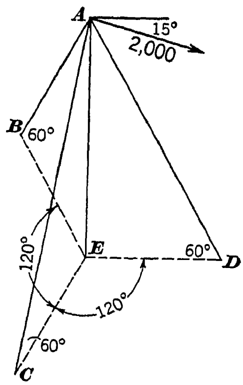


FIG. 232

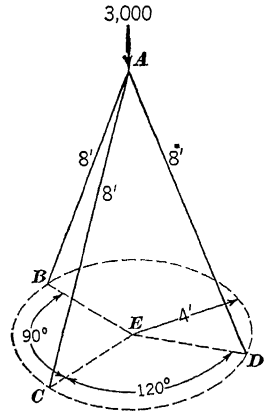


FIG. 233

224. If in Fig. 230 the plane ABC leans to the right at 60° with the horizontal and the 2,000-lb load is still horizontal, what are the stresses in the members AB, AC, and AD?

225. Determine the load carried by each leg of the tripod in Fig. 233.

63. Resultant of a Non-Coplanar, Non-Concurrent Force System.—This type of force system is generally reduced to a resultant force passing through some selected point and a resultant couple. Sometimes it is sufficient to express the turning effect of the system in terms of three couples ΣM_x, ΣM_y, and ΣM_z about the coordinate axes through the selected point.

Select some point as the center of coordinates, and draw the X, Y, and Z axes through this point.

By the method described in Art. 29, each force of the system may be transformed into an equal and parallel force passing through the center of coordinates and a couple which lies in the plane of the two parallel forces.

The original system has thus been transformed into a system of forces exactly equal and parallel to the original forces, but concurrent at the origin, and in addition a group of couples.

The resultant force R of the concurrent system of forces can be obtained in the manner explained in Art. 61.

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\alpha = \cos^{-1} \frac{\Sigma F_x}{R};\ \beta = \cos^{-1} \frac{\Sigma F_y}{R};\ \gamma = \cos^{-1} \frac{\Sigma F_z}{R}$$

The resultant of the system of couples may be found by extending the method of Art. 58, but it is much more convenient to use the following procedure.

Since the original system of forces is equivalent to the newly formed concurrent system plus the group of couples, the resultant turning moment about each of the coordinate axes may be determined by computing the moment of the original system of forces about each coordinate axis. In this manner the group of couples is reduced to three couples ΣM_x, ΣM_y, and ΣM_z. The resultant of these three couples may now be found from the relation

$$M_R = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2}$$

The angles which the vector representing the resultant moment M_R makes with the coordinate axes are given by the following expressions:

$$\alpha' = \cos^{-1} \frac{\Sigma M_x}{M_R};\ \beta' = \cos^{-1} \frac{\Sigma M_y}{M_R};\ \gamma' = \cos^{-1} \frac{\Sigma M_z}{M_R}$$

EXAMPLE 1

Determine the value of the resultant force acting through O and a couple for the force system shown in Fig. 234. The distances are marked off in feet.

Select a point on the line of action of each force, and resolve each force into its X, Y, and Z components at this point.

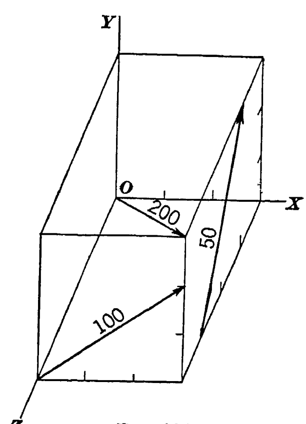


FIG. 234

F	F_x	F_y	F_z
50	0	$50\times\frac{3}{3.6} = 41.7$	$-50\times\frac{2}{3.6} = -27.8$
100	$100\times\frac{3}{3.6} = 83.3$	$100\times\frac{2}{3.6} = 55.5$	0
200	$200\times\frac{3}{5.83} = 102.8$	$200\times\frac{3}{5.83} = 102.8$	$200\times\frac{4}{5.83} = 137$
	$\Sigma F_x = 186.3$	$\Sigma F_y = 200$	$\Sigma F_z = 109.2$

$$R = \sqrt{186.3^2+200^2+109.2^2} = 294.2 \text{ lb}$$

$$\cos\alpha = \frac{186.3}{294.2};\quad \cos\beta = \frac{200}{294.2};\quad \cos\gamma = \frac{109.2}{294.2}$$

$$\alpha = 50.7^\circ \qquad \beta = 47.2^\circ \qquad \gamma = 68.2^\circ$$

	M_x	M_y	M_z
50	$-41.7\times 3 = -125.1$	$27.8\times 3 = 83.4$	$41.7\times 3 = 125.1$
100	$-55.5\times 4 = -222$	$83.3\times 4 = 333.2$	0
200	0	0	0
	$\Sigma M_x = -347.1$	$\Sigma M_y = 416.6$	$\Sigma M_z = 125.1$

$$M_R = \sqrt{347.1^2+416.6^2+125.1^2} = 556 \text{ ft-lb}$$

$$\cos\alpha' = \frac{347.1}{556};\quad \cos\beta' = \frac{416.6}{556};\quad \cos\gamma' = \frac{125.1}{556}$$

$$\alpha' = 128.7^\circ \qquad \beta' = 318.5^\circ \qquad \gamma' = 283^\circ$$

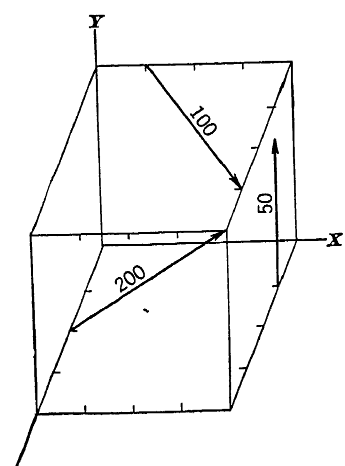


FIG. 235

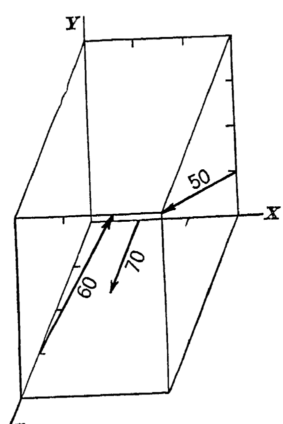


FIG. 236

PROBLEMS

226. Solve for the value of the resultant force passing through the origin and a couple for the force system shown in Fig. 235. The distances are marked off in feet. *Ans. 306.4 lb; 215 ft-lb.*

227. Reduce the force system shown in Fig. 236 to a single force passing through the origin and a couple. The distances are marked off in feet.

64. Equilibrium of a Non-Coplanar, Non-Concurrent Force System.—Since a non-coplanar, non-concurrent force system can be reduced to a single resultant force and a couple, a system of this type that is to be in equilibrium must have the following characteristics: The resultant force must equal zero, or $R=0$; also the resultant moment must be zero, or $M_R=0$.

The usual method of solution for a system of this type is to project the system onto each of the coordinate planes in turn. These projections are usually coplanar, non-concurrent systems. The conditions of equilibrium for each of the projections are: $\Sigma F_h=0$, $\Sigma F_v=0$, and $\Sigma M=0$. Each of the projections can then be solved by the methods developed for coplanar force systems.

Each projection furnishes two summation equations and a moment equation. If these equations are written with reference to the ordinary coordinate axes, they will be as follows:

$$\begin{array}{lll} \Sigma F_x=0 & \Sigma F_y=0 & \Sigma M_z=0 \\ \Sigma F_y=0 & \Sigma F_z=0 & \Sigma M_x=0 \\ \Sigma F_x=0 & \Sigma F_z=0 & \Sigma M_y=0 \end{array}$$

If the duplicates are eliminated from the above group of equations, the independent equations remaining are:

$$\begin{array}{lll} \Sigma F_x=0 & \Sigma F_y=0 & \Sigma F_z=0 \\ \Sigma M_x=0 & \Sigma M_y=0 & \Sigma M_z=0 \end{array}$$

Thus, six unknown quantities can be solved for in a non-coplanar, non-concurrent force system.

EXAMPLE 1

If the hoisting engine shown diagrammatically in Fig. 237 is to be run at a uniform speed, determine the value of the resultant force Q which must be applied to the piston for the position shown. Also determine the X, Y, and Z components of the bearing reactions when the crankpin is in the position shown. The connecting-rod is at an angle of 15° with the horizontal, and the crank is at 60°.

The radius of the crank is 1 ft and the diameter of the drum is 1 ft. The weight of the drum is 3,000 lb and that of the flywheel 2,000 lb.

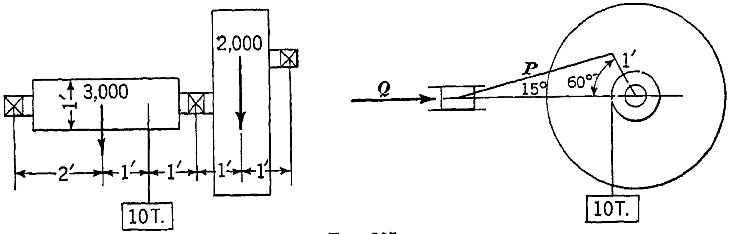


FIG. 237

Project the force system onto each of the coordinate planes. Each projection is a coplanar, non-concurrent system, for which the conditions of equilibrium are:

$$\Sigma F_h = 0;\ \Sigma F_v = 0;\ \Sigma M = 0$$

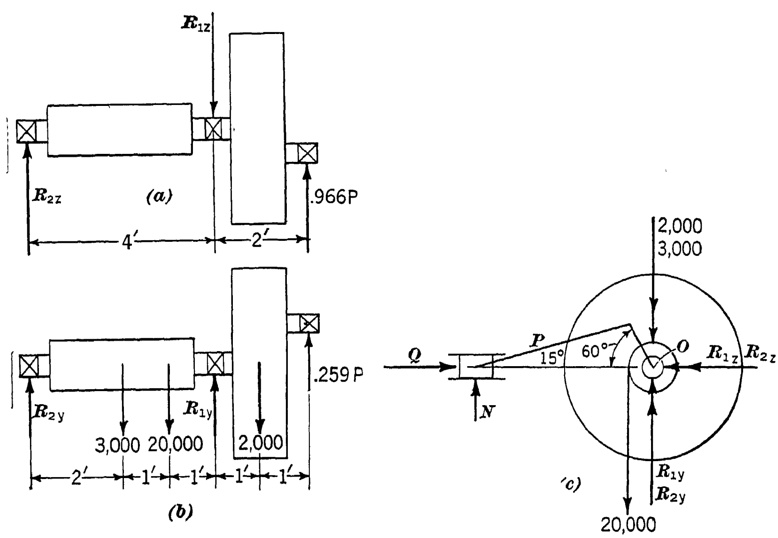


FIG. 238

In Fig. 238 (*c*), which is the *YZ* projection, there are seven unknown forces. If only the drum, shaft, and flywheel are taken as the free body, there are five unknown forces, four of which pass through the center *O* of the shaft. This free body may be solved for the force *P* by taking moments about *O*:

$$\Sigma M_O = 0$$
$$P \times 0.966 \times 1 - 20{,}000 \times 0.5 = 0$$
$$P = 10{,}352 \text{ lb, C.}$$

If the cross-head is used as the free body, the following relation may be applied:

$$\Sigma F_h = 0$$
$$Q - 0.966 \times 10{,}352 = 0$$
$$Q = 10{,}000 \text{ lb}$$

From Fig. 238 (*a*):

$$\Sigma M_{R2z} = 0$$
$$-R_{1z} \times 4 + 0.966 \times 10{,}352 \times 6 = 0$$
$$R_{1z} = 15{,}000 \text{ lb}$$

$$\Sigma M_{R1z} = 0$$
$$-R_{2z} \times 4 + 0.966 \times 10{,}352 \times 2 = 0$$
$$R_{2z} = 5{,}000 \text{ lb}$$

From Fig. 238 (*b*):

$$\Sigma M_{R2y} = 0$$
$$4R_{1y} - 3{,}000 \times 2 - 20{,}000 \times 3 - 2{,}000 \times 5 + 0.259 \times 10{,}352 \times 6 = 0$$
$$R_{1y} = 14{,}978 \text{ lb}$$

$$\Sigma M_{R1y} = 0$$
$$-4R_{2y} + 3{,}000 \times 2 + 20{,}000 \times 1 - 2{,}000 \times 1 + 0.259 \times 10{,}352 \times 2 = 0$$
$$R_{2y} = 7{,}340 \text{ lb}$$

From Fig. 238 (*a*) or (*b*), R_{1x} and R_{2x} are zero, since there are no forces acting in the X direction.

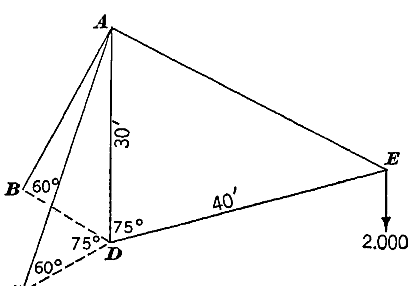


Fig. 239

EXAMPLE 2

In the derrick shown in Fig. 239 assume that the boom can swing through 360°. Place the boom in the position which will

cause maximum tension in AB. With the boom in this position, solve for the stresses in AB, AC, and AD.

The solution of this problem involves two new ideas, the determination of the "overturning force" and the position of the boom which will cause the maximum stress in a given back-stay.

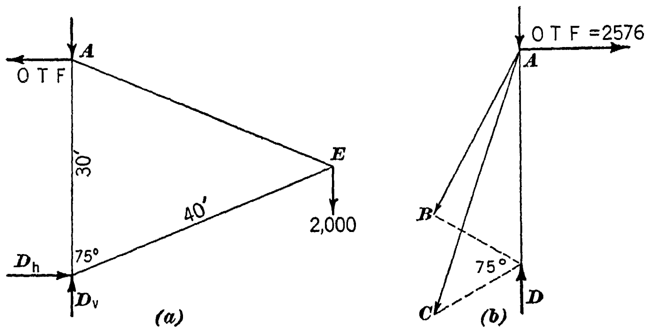


FIG. 240

In Fig. 240 (a), the post and boom are shown as a free body with the "overturning force" OTF acting at A. With a line through D as the axis of moments, the value of the force OTF can be determined in the following manner:

$$\Sigma M_D = 0$$
$$30 \text{ OTF} - 2{,}000 \times 40 \times 0.966 = 0$$
$$\text{OTF} = 2{,}576 \text{ lb}$$

In Fig. 240 (b), the post and two members are shown with the force OTF reversed acting at A. The force OTF is the horizontal effect produced at A by the 2,000-lb load. The forces acting at A form a concurrent system. If these forces are projected on a horizontal plane through A, Fig. 241 (a) is obtained. If the force OTF is placed so that it is at 90° with the plane of the guy AC, the stress in AB will be maximum. The proof of this statement is left to the student.

In Fig. 241 (b) is shown the force triangle for the three forces acting at A in Fig. 241 (a).

$$\frac{0.5\ AB}{\sin 90°} = \frac{0.5\ AC}{\sin 15°} = \frac{2{,}576}{\sin 75°}$$

$$AB = 5{,}330 \text{ lb, T. and } AC = 1{,}380 \text{ lb, C.}$$

In the force triangle of Fig. 241 (*b*), it will be noted that the direction of AB is away from the point A and that AC is toward A. Therefore, the stress in AB is tension, and that in AC is compression.

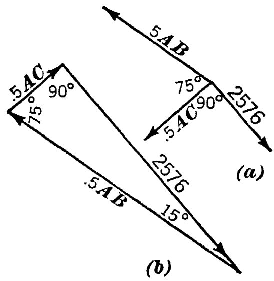


FIG. 241

If the pin E is taken as a free body, the stresses in AE and DE can be obtained.

$$AE = 2{,}885 \text{ lb, T. and } DE = 2{,}667 \text{ lb, C.}$$

The compression in the post must balance the vertical components of AB, AC, and AE.

$$AD = 5{,}334 \times 0.866 - 1{,}380 \times 0.866 + 2{,}885 \times \frac{19.66}{43.3}$$

$$AD = 4{,}735 \text{ lb, C.}$$

PROBLEMS

228. In Example 1, move the crank 90° in a counter-clockwise direction. Determine the stress in the connecting-rod and the amounts and directions of the Y and Z components of the reactions. *Ans.* $P = 16{,}030$ *lb;* $R_{1y} = 15{,}407$ *lb;* $R_{2z} = 7{,}910$ *lb;* $R_{2y} = 7{,}198$ *lb;* $R_{1z} = 23{,}760$ *lb.*

229. In Example 2, let the plane of the boom bisect the angle BDC. Compute the stresses in AB, AC, and the post. If the boom is turned through 180°, what stress will the post carry?

230. Determine the maximum stress in AC, Fig. 242, and also the stress in AB when the stress in AC is maximum.

231. If in Fig. 242 the point B is 4 ft directly above its present position and boom GFD bisects the 120° angle, what are the stresses in members AB and AC?

232. Compute the maximum compressive stress which may occur in member AB, Fig. 242.

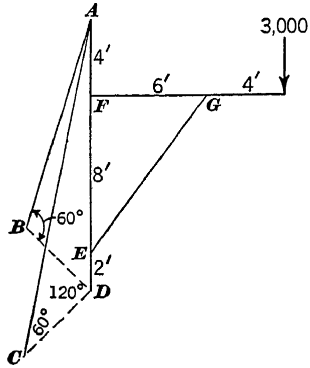


FIG. 242

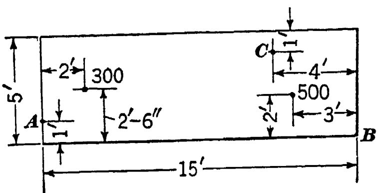


FIG. 243

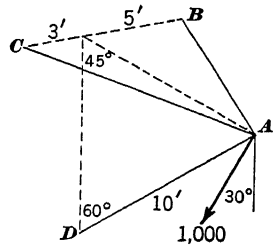


FIG. 244

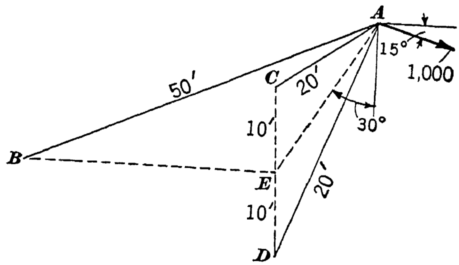


FIG. 245

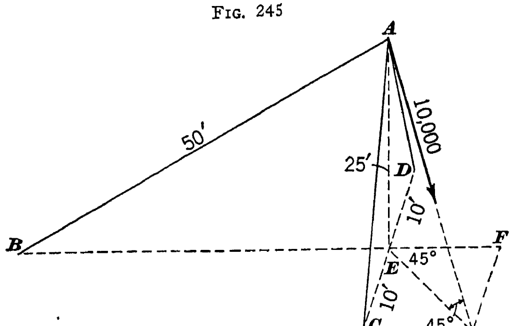


FIG. 246

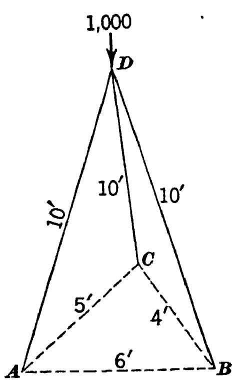


FIG. 247

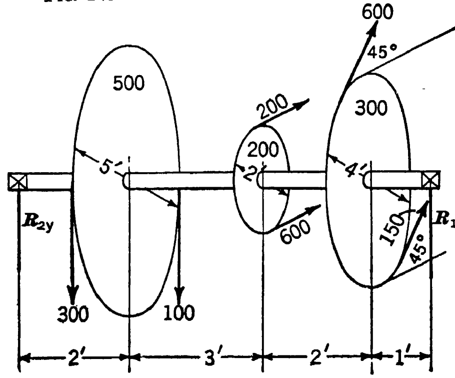


FIG. 248

REVIEW PROBLEMS

233. The platform shown in Fig. 243 is supported by ropes attached at points A, B, and C. It carries loads of 500 and 300 lb as shown. What is the stress in each rope? *Ans.* $A = 261$ *lb;* $B = 167$ *lb;* $C = 372$ *lb.*

234. What are the stresses in the members AB, AC, and AD in Fig. 244?

235. Compute the stresses in AB, AC, and AD in Fig. 245. The ,000-lb force is in the plane ABE and is 15° below the horizontal.

236. In Fig. 246, the 10,000-lb force acts 45° below the horizontal and in a plane making an angle of 45° with the plane ABE. Determine the stresses in AB, AC, and AD.

237. In Fig. 247 is shown a tripod with legs 10 ft long resting on a triangular base with sides of 4, 5, and 6 ft. Determine the amount of compression in each leg.[1] *Ans.* $DA = 440$ *lb;* $DB = 450$ *lb;* $DC = 120$ *lb.*

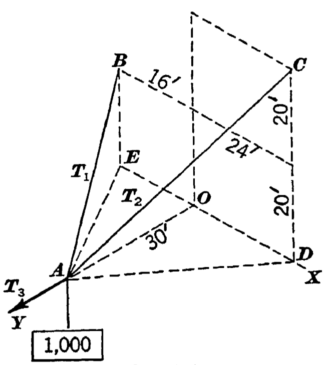


FIG. 249

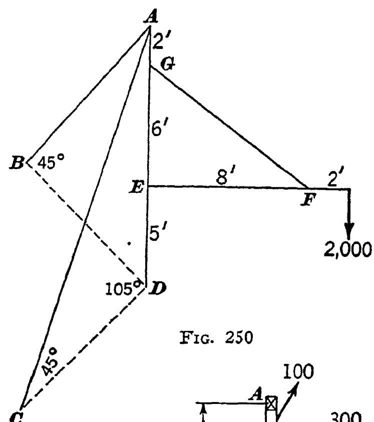


FIG. 250

238. Determine the components of the bearing reactions for the line shaft shown in Fig. 248.

239. Determine the tensions T_1, T_2, and T_3 in the three cords in Fig. 249.

240. Place the boom in the position which will produce the maximum stress in AC, Fig. 250. Compute the stresses in members AC, AB, and FG.

241. The shaft in Fig. 251 turns at a constant speed. If the shaft and the pulleys weigh 400 lb, what are the X, Y, and Z components of the bearing reactions at A and B?

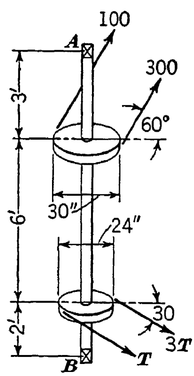


FIG. 251

242. A circular plate 6 ft in diameter is supported by three equal-length wires which are attached to the circumference at points A, B, and C. Points

[1] The ratio of any side of a triangle to the sine of the opposite angle equals 2 times the radius of the circumscribed circle.

A and B are 120° apart. A 300-lb load is eccentric 1 ft on the radius to point A. Determine the angle between the radii drawn to A and C for equal loads on all wires.

243. Solve Problem 237 if AD is 9.5 ft and BD is 9 ft. Solve graphically.

244. In Fig. 252 points D, E, F, and G are all in the plane of the ground. Point B is 2 ft vertically above E, and point C is 3 ft vertically above F. Solve for the stresses in the legs AB, AC, and AD of the tripod caused by the 5,000-lb load.

245. Solve for the stresses in EG, AB, and AC of Fig. 253. The plane of the boom EFG is 15° back of the vertical plane ADX.

246. If the boom in Fig. 253 can be moved through 360°, determine the maximum compressive stresses which members AB and AC will be called upon to resist.

247. If the point C is 5 ft vertically above the position shown in Fig. 253, what are the stresses in members AB and AC?

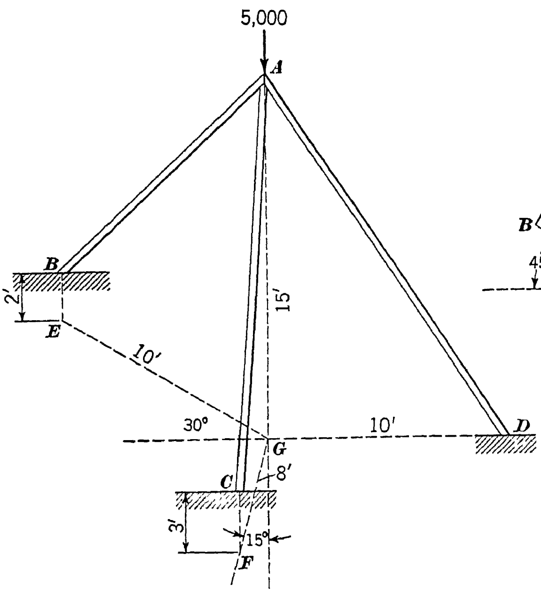


FIG. 252

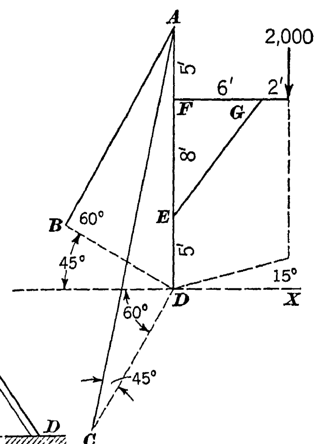


FIG. 253

CHAPTER 8

FLEXIBLE CABLES

65. Classes of Cables.—Cables are usually divided into two general classes, according to the manner in which the load is applied to the cable:

(*a*) Cables for which the load may be considered as uniformly distributed over the horizontal distance between the points of support. This type of loading causes the cable to take the form of a parabola.

(*b*) Cables for which the load must be considered as uniformly distributed along the curve formed by the cable. A cable loaded in this manner will take the form of a catenary.

Under class (*a*) can usually be placed such cables as the main supporting cables of suspension bridges, carrier or messenger cables which are used to support a heavy trolley or telephone cable, and also certain transmission line cables. This class includes any cable whose points of support are sufficiently close together to permit the sag of the cable to be small.

Those cables which have a large sag in proportion to the span must be placed in class (*b*).

If the sag does not exceed one-tenth of the span, the parabola method is generally the method used. It will give results which are at least as accurate as the experimental data involved in the solution.

66. Parabola Method.—In Fig. 254 (*a*), let ABC represent a telegraph wire supported at the points A and B, both of which are at the same elevation and a distance l apart. If the sag d is small in comparison with the horizontal distance AB, the weight of the cable can be assumed to be uniformly distributed over the distance AB without introducing a large error.

At the center of the span the direction of the tension in the cable is horizontal. At any other point the direction of the tension is given by the tangent to the curve formed by the cable. In Fig. 254 (*b*) half of the cable is shown as a free body, with the horizontal tension at the center indicated by H and the resultant tension at the support B indicated by T. The only other force acting

on this section of the cable is the weight of the cable, which can be considered as acting at a distance $\frac{l}{4}$ from the point B, when the weight is assumed to be uniformly distributed over the distance between supports. Since the free body is held in equilibrium by three forces, these three forces must pass through a common point D.

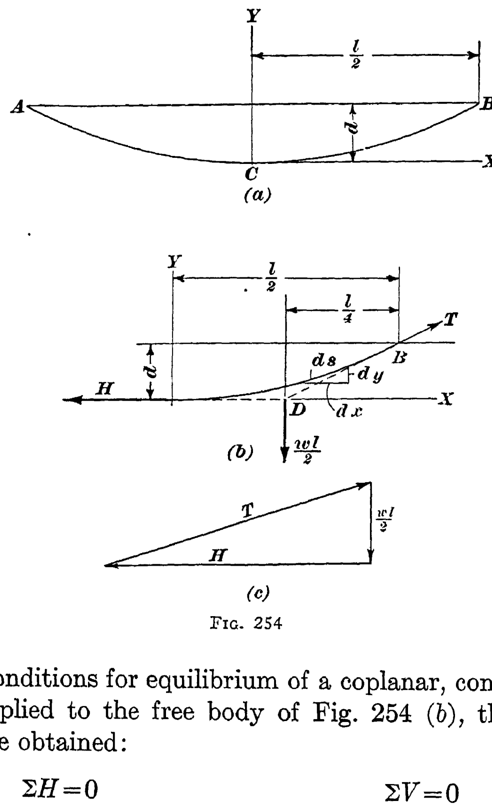


FIG. 254

If the conditions for equilibrium of a coplanar, concurrent system are applied to the free body of Fig. 254 (*b*), the following relations are obtained:

$$\Sigma H = 0 \qquad \Sigma V = 0$$

$$T_h = H \qquad T_v = \frac{w\,l}{2}$$

$$T = \sqrt{H^2 + \left(\frac{w\,l}{2}\right)^2} \tag{1}$$

$$\Sigma M_B = 0$$

$$H\,d = \frac{w\,l}{2}\,\frac{l}{4}$$

$$d=\frac{w\,l^2}{8\,H} \tag{2}$$

$$H=\frac{w\,l^2}{8\,d} \tag{3}$$

Equations (2) and (3) are the equations of a parabola with its vertex at C and its axis vertical. The value of T may now be found in terms of the sag and span by eliminating H from equations (1) and (3). Thus,

$$T=\frac{w\,l}{2}\sqrt{1+\frac{l^2}{16\,d^2}}$$

For cases in which the sag is less than 5 per cent of the span, it is generally permissible to assume that $H=T$.

The length s of the cable may be determined in the following manner. If ds represents a differential length of the curve, then:

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$2\int_0^{\frac{s}{2}} ds=2\int_0^{\frac{l}{2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

If in equation (2) $d=y$ and $l=2\,x$, then $y=\frac{w\,x^2}{2\,H}$ and $\frac{dy}{dx}=\frac{w\,x}{H}$.

Hence,

$$s=2\int_0^{\frac{l}{2}}\sqrt{1+\left(\frac{w\,x}{H}\right)^2}\,dx \tag{4}$$

Since the sag of the cable is small when compared with the span, the slope $\frac{dy}{dx}$ will be small. Therefore, $\frac{dy}{dx}=\frac{w\,x}{H}$ will be a very small quantity. It is then possible to write equation (4) as follows, without introducing much error.

$$s=2\int_0^{\frac{l}{2}}\left[1+\frac{1}{2}\left(\frac{w\,x}{H}\right)^2\right]dx$$

$$s=l+\frac{w^2\,l^3}{24\,H^2} \tag{5}$$

By combining equation (5) with equation (3),

$$s=l+\frac{8\,d^2}{3\,l} \tag{6}$$

If more accurate results are desired, the radical of equation (4) may be expanded into a series by substitution in the following equation:

$$(a+z)^n=a^n+n\,a^{n-1}\,z+n(n-1)a^{n-2}\,z^2 \quad . \quad .$$

Thus,

$$s=l+\frac{w^2\,l^3}{24\,H^2}-\frac{w^4\,l^5}{640\,H^4}$$

or

$$s=l+\frac{8\,d^2}{3\,l}-\frac{32\,d^4}{5\,l^3}$$

EXAMPLE

Determine the tension in a steel cable, Fig. 255, which weighs 0.5 lb per ft, if the distance between the supports is 600 ft and the sag at the center is 15 ft. What length of cable will be required?

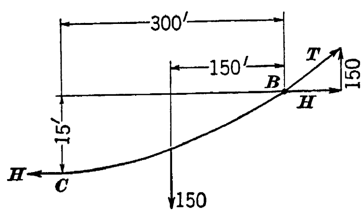


FIG. 255

$$\Sigma M_B=0$$
$$15\,H-150\times150=0$$
$$H=1{,}500\text{ lb}$$

$$T=\sqrt{150^2+1{,}500^2}=1{,}508\text{ lb}$$
$$s=600+\frac{8\times15^2}{3\times600}-\frac{32\times15^4}{5\times600^3}$$
$$s=600+1-0.0015=600.9985\text{ ft}$$

It will be observed that the last term in the above equation has little practical significance. The solution of this problem by the

catenary method will be found in Art. 68. Comparison of the results obtained by the two methods shows a variation which is well within the limits of error introduced by the experimental data on strength of materials. This comparison clearly shows that for small sags the parabola method is sufficiently accurate. Attention is also called to the small difference between T and H.

PROBLEMS

248. If a copper cable weighs 0.465 lb per ft, the distance between supports is 500 ft, and the maximum allowable pull which the cable can carry is 3,200 lb, what is the allowable sag? What length of wire will be required? *Ans. 4.54 ft; 500.11 ft.*

249. If the allowable tensile stress for copper is 12,000 psi, what is the maximum span for a copper cable $\frac{1}{2}$ in. in diameter? The sag is to be $\frac{1}{20}$ of the span, and copper weighs 556 lb per cu ft.

250. A transmission line has two towers 75 ft high spaced 300 ft apart. There are three $\frac{3}{8}$-in. copper cables on each tower, each of which has a sag of 10 ft at the center of the span. Copper weighs 556 lb per cu ft. Assume that all three cables are attached to the towers at the top. Determine the bending moment at the base of each tower and the stress in each cable.

251. A wire weighing 0.3 lb per ft is supported at two points 120 ft apart. If the maximum pull permitted in the wire is 800 lb, what sag will the wire have? What length of wire will be needed?

67. Supports at Different Levels.—In many power transmission lines and other structures involving cables, it is necessary that the cable supports be placed at different levels. This necessitates the solution of two free bodies.

EXAMPLE

The cable shown in Fig. 256 (a) carries a load of 10 lb per foot of span. Determine the total pull in the cable at points A, B, and C.

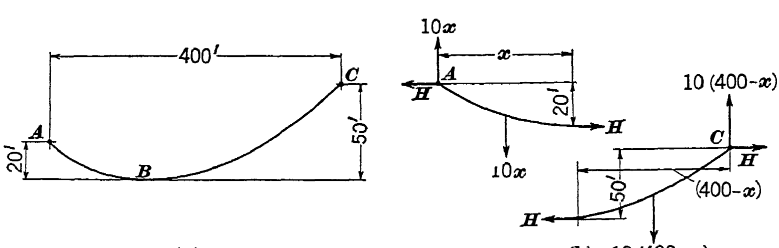


FIG. 256

Using the free bodies shown in Fig. 256 (*b*), we may write:

$$\Sigma M_A=0 \qquad\qquad \Sigma M_C=0$$

$$20H-\frac{10x^2}{2}=0 \qquad\qquad 50H-\frac{10(400-x)^2}{2}=0$$

$$H=0.25x^2 \qquad\qquad H=\frac{160{,}000-800x+x^2}{10}$$

$$2.5x^2=160{,}000-800x+x^2$$

$$x=155.1 \text{ ft}$$

$$H=6{,}013.7 \text{ lb}$$

$$T_A=\sqrt{6{,}013^2+1{,}551^2} \qquad\qquad T_C=\sqrt{6{,}013^2+2{,}449^2}$$

$$T_A=6{,}212 \text{ lb} \qquad\qquad T_C=6{,}494 \text{ lb}$$

PROBLEMS

252. If the cable shown in Fig. 257 carries a load of 2,000 lb per horizontal foot, what are the total pulls at points A, B, C, and D? Solve for the distance X. *Ans.* $T_A=53{,}850$ *lb;* $T_B=50{,}000$ *lb;* $T_C=53{,}850$ *lb;* $T_D=57{,}405$ *lb.*

253. The cable shown in Fig. 258 weighs 0.3 lb per foot of horizontal span. What are the total tensions at points A, B, and D, and also the clearance distance CE?

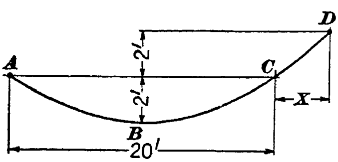


FIG. 257

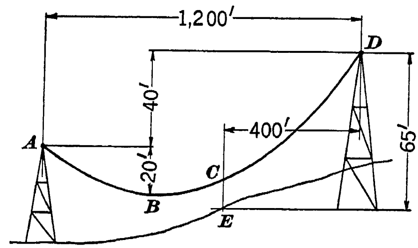


FIG. 258

68. Catenary.—For cables having a sag so large that the parabola solution will not produce sufficiently accurate results, the catenary method must be used. See Fig. 259 (*a*).

In the following development of the catenary relationships, let

w = weight per unit length of cable;

T = resultant tension in the cable at a point at distance s from the lowest point on the curve;

H = total tension in the cable at the lowest point on the curve;

c = the distance from the lowest point on the curve to the directrix.

The directrix is so located that $H=w\,c$.

The conditions of equilibrium may be applied to the free body shown in Fig. 259 (*b*).

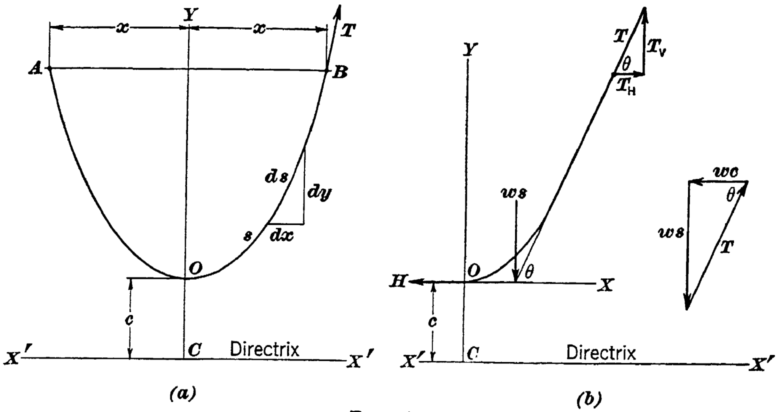


FIG. 259

$$\Sigma F_v = 0 \qquad\qquad \Sigma F_h = 0$$

$$T_V = w\ s \qquad\qquad T_H = H = w\ c$$

$$\tan\theta = \frac{dy}{dx} = \frac{T_V}{T_H} = \frac{w\ s}{w\ c}$$

$$\frac{dy}{dx} = \frac{s}{c} \tag{1}$$

$$(dx)^2 = (ds)^2 - (dy)^2 = (ds)^2 - \frac{s^2}{c^2}\,(dx)^2$$

$$dx = \frac{c\ ds}{\sqrt{c^2+s^2}} \tag{2}$$

$$\int_0^x dx = c\int_0^s \frac{ds}{\sqrt{c^2+s^2}}$$

$$x = c\log_e \frac{s+\sqrt{c^2+s^2}}{c} \tag{3}$$

Equation (3) may be converted to the following exponential form:

$$e^{\frac{x}{c}} = \frac{s+\sqrt{c^2+s^2}}{c}$$

When this equation is solved for s, the result is

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \tag{4}$$

If the value of s from equation (4) is substituted in equation (1), the result is

$$dy=\frac{1}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right)dx \tag{5}$$

If the origin is now shifted from O to C and equation (5) is integrated with the limits of y taken as c and y,

$$\int_c^y dy=\frac{c}{2}\left(\int_0^x \frac{1}{c}\,e^{\frac{x}{c}}\,dx-\int_0^x \frac{1}{c}\,e^{-\frac{x}{c}}\,dx\right)$$

$$y-c=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)-c$$

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) \tag{6}$$

Equations (4) and (6) may be squared and subtracted. Then,

$$y^2=c^2+s^2 \tag{7}$$

Combine equations (7) and (3) to obtain the relation

$$x=c\log_e\frac{s+y}{c} \tag{8}$$

From the force triangle in Fig. 259 (b),

$$T^2=w^2\,c^2+w^2\,s^2=w^2(c^2+s^2)=w^2\,y^2$$

or

$$T=w\,y \tag{9}$$

From the equations which have just been developed the following useful relationships are obtained.

1. The horizontal component of the tension in the cable is constant for all points and is

$$T_H=H=w\,c$$

2. The vertical component of the tension is a variable quantity and depends on the distance along the curve from the lowest point of the curve. Thus,

$$T_V=w\,s$$

3. The resultant tension at any point along the curve is

$$T=w\,y$$

4. From the logarithmic equation $x = c \log_e \frac{s+y}{c}$, the horizontal distance between points of support can be obtained by trial. The length of half the span is taken as x.

EXAMPLE 1

The example of Art. 66 will now be solved by the catenary method.

$$y = c + 15$$

From equation (7),

$$y^2 = c^2 + s^2$$
$$c^2 + 30c + 225 = c^2 + s^2$$
$$s = \sqrt{30c + 225}$$

Let $c = 3{,}000$. Then,

$$x = c \log_e \frac{s+y}{c} = c \log_e \frac{\sqrt{30 \times 3{,}000 + 225} + 3{,}000 + 15}{3{,}000}$$
$$x = 300 \text{ ft}$$
$$300 = 3{,}000 \log_e \frac{300.35 + 3{,}015}{3{,}000} = 3{,}000 \log_e 1.105$$
$$300 = 3{,}000 \times 0.0998 = 299.4$$
$$s^2 = y^2 - c^2 = 3{,}015^2 - 3{,}000^2 = 90{,}225$$
$$s = 300.37 \text{ and } 2s = 600.74 \text{ ft}$$
$$T = wy = 0.5 \times 3{,}015 = 1{,}507.5 \text{ lb}$$

Compare the results just obtained with those given by the parabola method used in Art. 66. It is evident that the solution of this problem by the parabola method is entirely satisfactory.

EXAMPLE 2

A cable weighs 2 lb to the foot and has a span of 500 ft. If the allowable tension is 2,000 lb, what are the sag and length of the cable?

$$T = wy \text{ and } y = \frac{2{,}000}{2} = 1{,}000$$

$$y^2 = c^2 + s^2 \text{ and } s = \sqrt{y^2 - c^2}$$

$$x = c \log_e \frac{s+y}{c} = c \log_e \frac{\sqrt{y^2 - c^2} + y}{c}$$

$$\text{Try } c=968;\quad 250=968\ \log_e 1.292=247.8$$
$$c=967;\quad 250=967\ \log_e 1.297=251.4$$
$$c=967.3;\ 250=967.3\ \log_e 1.295=250.04\ \text{(satisfactory)}$$

$$s=\sqrt{1{,}000^2-967.3^2}=253.4\text{ ft};\ 2s=506.8\text{ ft}=\text{length of cable}$$
$$\text{sag}=y-c=1{,}000-967.3=32.7\text{ ft}$$

PROBLEMS

254. A cable 500 ft long weighs 2 lb per ft and is supported at two points on the same level. If the allowable tension is 2,000 lb, what are the span and the sag? *Ans. 494.78 ft; 31.76 ft.*

255. A cable, weighing 1.5 lb per ft, is supported at two points at the same elevation and 800 ft apart. The sag at the center of the span is 200 ft. What are the maximum tension in the cable, and the length of cable required? What percentage of error would be introduced into the tension if the parabola solution were used?

69. Catenary Solution When Supports Are at Different Elevations.—Many power transmission lines and cableways must have their supports placed at different elevations, because of the local surface conditions. A problem of this type is illustrated in Fig. 260.

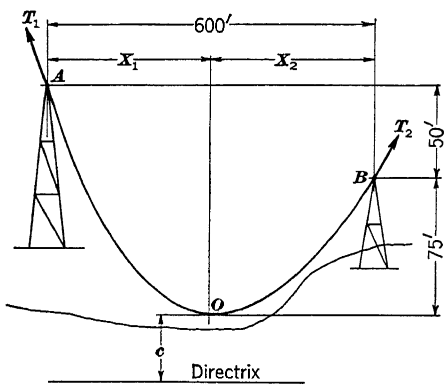


Fig. 260

The cable shown weighs 3 lb per ft. What total tension will be set up at each support, and what is the required length of the cable?

The method of solution is as follows. Divide the cable into two parts at the point O. Each part is then taken as a free body in equilibrium.

$$600 = x_1 + x_2;\; y_1 = c + 125;\; y_2 = c + 75$$

$$y_1^2 = c^2 + s_1^2 \text{ and } y_2^2 = c^2 + s_2^2$$

$$s_1 = \sqrt{250c + 15{,}625} \text{ and } s_2 = \sqrt{150c + 5{,}625}$$

$$600 = c \log_e \frac{s_1 + y_1}{c} + c \log_e \frac{s_2 + y_2}{c}$$

$$600 = c \log_e \frac{\sqrt{250c + 15{,}625} + c + 125}{c} + c \log_e \frac{\sqrt{150c + 5{,}625} + c + 75}{c}$$

Solve the above equation for c by trial.

For $c = 480$,

$600 = 480 \log_e 2.026 + 480 \log_e 1.736$

$600 = 603.6$

For $c = 475$,

$600 = 475 \log_e 2.035 + 475 \log_e 1.74$

$600 = 600.54$, which is satisfactory

$s_1 = \sqrt{250 \times 475 + 15{,}625}$ $\quad s_2 = \sqrt{150 \times 475 + 5{,}625}$

$s_1 = 366.5$ $\quad s_2 = 277.08$

$s = 366.5 + 277.08 = 643.58$ ft

$T_1 = wy = 3(475 + 125) = 1{,}800$ lb

$T_2 = 3(475 + 75) = 1{,}650$ lb

CHAPTER 9

FRICTION

70. Nature of Friction.—Friction is generally regarded as a destroyer of energy and as something which should be avoided wherever possible. This view is not entirely true. There are many machines and mechanical devices, such as brakes, friction clutches, and belt drives, which depend on friction for the transfer of energy from one unit to another. Many ordinary operations, such as walking, jumping, and the movement of trains and motor driven vehicles, could not be accomplished if frictional resistance were not present at certain contacting surfaces.

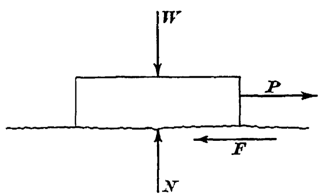


FIG. 261

When two surfaces are in contact, the motion of the surfaces parallel to each other is resisted by friction and certain other attractive forces, such as molecular adhesion. *The magnitude of the frictional resistance varies directly with the normal pressure and is independent of the area of contact.* Adhesion depends on the area of contact and is independent of the pressure.

Since the resistance to motion must be determined experimentally, it is impossible to determine definitely the proportion of the resistance which is due to friction and to each of the other causes. Since the surfaces of the materials used in engineering are usually comparatively rough, the assumption is made that all resistance to motion is due to friction. The frictional resistance is caused by the interlocking of the irregularities of the rough surfaces while in contact, as is indicated in Fig. 261.

71. Plane Friction.—Think of the block shown in Fig. 261 as resting on any ordinary horizontal surface. The term ordinary surface is here used to designate any surface which is not theoretically smooth.

If a small force P is applied to the block horizontally, or parallel to the plane, there will be no motion. A frictional resistance $F = P$ is developed at the surfaces of contact between the block and the

plane. Since this force F is just large enough to balance P, no motion will result. If P is gradually increased, F will also increase a like amount; and the relation $F=P$ still remains true. After F has increased to a certain limiting or maximum value, which depends on the normal pressure between the block and the plane and on the nature of the surfaces of contact, no further increase of F can take place. If P continues to increase, becoming greater than this limiting value of the frictional resistance, designated as F', the equilibrium is destroyed and the block moves because of the action of the unbalanced part of the force P.

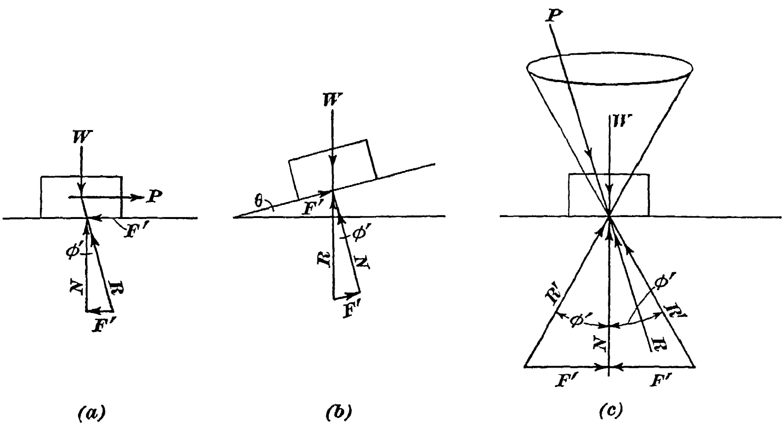


FIG. 262

After motion begins, the frictional resistance decreases. This decreased frictional force, which would be developed during uniform motion, is known as the kinetic frictional force. *The frictional force always acts parallel to the surfaces in contact, and is so directed as to oppose the motion of the surfaces relative to each other.*

72. **Coefficient of Friction, Angle of Repose, and Cone of Friction.**—Assume that the force P which is acting on the block shown in Fig. 262 (a) is just great enough to cause motion to the right to be impending.

The block is then in equilibrium under the action of the four forces P, F', W, and N. The forces F' and N may be combined by the parallelogram or triangle method into their resultant R. The force R is the resultant effect of the supporting plane on the block.

The angle ϕ' which R makes with N is called the angle of friction. The tangent of the angle ϕ' is $\frac{F'}{N}$. The ratio $\frac{F'}{N}$ is also known as the coefficient of static friction. Thus,

$$\tan \phi' = \frac{F'}{N} = f' = \text{the coefficient of static friction}$$

The coefficient of kinetic friction is given by the ratio $\frac{F}{N}$, the force F being the frictional force developed between two surfaces which are moving relative to each other. The coefficient of kinetic friction is always less than the coefficient of static friction.

$$f = \frac{F}{N} = \text{the coefficient of kinetic friction}$$

The forces F' and F may be determined experimentally by considering a block on a horizontal plane, as in Fig. 262 (*a*); or, if the block is placed as in Fig. 262 (*b*) on a plane inclined so that impending slipping of the block exists, then the angle θ is the angle of repose and $\tan \theta = \frac{F'}{N}$. However,

$$\tan \phi' = \frac{F'}{N} = f'$$

Therefore the angle of repose θ is equal to ϕ', which is the angle of limiting static friction. If the angle θ is adjusted so that the block moves down the plane at a constant velocity, then

$$\tan \theta = \frac{F}{N} = f = \text{the coefficient of kinetic friction}$$

If a cone is generated by revolving a line R' at the angle of limiting static friction ϕ' with N, the normal to the frictional surface, as in Fig. 262 (*c*), the cone generated is called the cone of friction. If P, the resultant of all external forces acting on the block except the reaction R of the plane, lies inside the cone generated by R', no motion can occur because the horizontal component of P will be less than the available limiting static friction F'. If P lies in the surface of the cone, the horizontal component of P is equal to the amount of the limiting static friction F' and motion is impending. If P extends outside the cone, the block will move.

73. Values of the Coefficient of Plane Friction.—The engineering handbooks and journals show a wide range of values for the coefficients of friction. This divergence is due, no doubt, to the variable conditions under which the investigators worked. It is therefore essential that engineers use extreme caution and good judgment in the selection of coefficients for any given set of conditions.

In the case of lubricated surfaces the values of the coefficients are much smaller. For well lubricated surfaces, the condition is more nearly that of two surfaces separated by a film of the lubricating medium. The frictional resistance under these conditions is that offered by the lubricant to shearing rather than by friction between the two surfaces. For information and the study of friction as applied to lubricated surfaces, the student is referred to the Proceedings of A.S.M.E. and other engineering societies.

The values for coefficients of static friction which are given in the table below are those given in most handbooks and are credited to the work done by Morin and Coulomb.

Coefficients of Friction for Dry Surfaces

Wood on wood	0.3 to 0.7
Wood on metal	0.2 to 0.6
Metal on metal	0.15 to 0.3
Leather on wood	0.25 to 0.5
Leather on metal	0.3 to 0.6

74. Laws of Friction.—The following general rules or laws of friction may be stated:

1. The limiting static frictional force and the kinetic frictional force are proportional to the normal pressure. Therefore, the coefficients of friction are independent of the normal pressure.

2. The frictional resistance is independent of the area of contact, and is directly proportional to the normal pressure.

3. For moderate speeds, friction is independent of the velocity.

4. The direction of the frictional resistance is *parallel to the surfaces in contact and is so directed as to oppose motion of the surfaces relative to each other.*

Rule 4 can be explained in the following manner: Consider a block A resting on a table B. If it is assumed that A is the free body, and it is moved to the right, the frictional resistance will act

to the left. *The frictional resistance opposes the motion of the block relative to the table.*

Next let the free body A stand still, and cause the table B to be moved to the right, under the block A. The table will tend to cause the block to move with it to the right. In this case, *the frictional force is to the right, or in the direction of the motion.*

The actions just explained may be summed up in two general statements:

(*a*) When the free body is in motion, or when motion is impending, the frictional force opposes the motion.

(*b*) When the free body is standing still, the frictional force acting on the free body is in the direction of the motion or impending motion of the moving surface.

If the student will thoroughly fix the significance of these two statements in his mind, he should have little difficulty in understanding plane friction.

75. **Motion Along a Plane.**—There are many examples of friction along a plane in such machines as planers and shapers, and in engine cross-heads and other similar mechanisms.

EXAMPLE 1

Determine the value of the force P, Fig. 263, acting parallel to the plane, that is necessary to cause the 100-lb weight to be just on the point of moving to the right. The coefficient of static friction is $f'=0.2$.

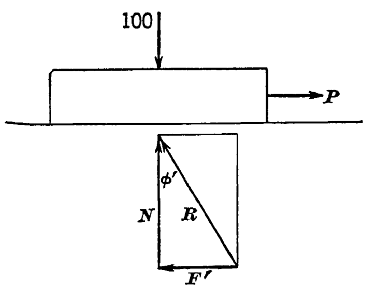


FIG. 263

The weight is in equilibrium under the action of the following four forces: 100-lb, P, N, and F', which acts parallel to the plane. Summations normal and parallel to the plane give the following equations.

ΣF perpendicular to the plane:

$$N-100=0$$
$$N=100 \text{ lb}$$

ΣF parallel to the plane:

$$P-F'=0$$
$$P=F'$$

By definition, $f'=\tan \phi'=\frac{F'}{N}$. Hence,

$$P=F'=100\times 0.2=20 \text{ lb}$$

EXAMPLE 2

If the 100-lb weight of Example 1 is placed on the 30° plane shown in Fig. 264, determine the value of the horizontal force P which will cause motion to the right to impend. The coefficient $f'=0.2$.

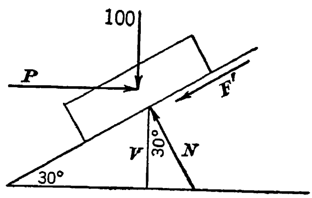


FIG. 264

If summations are made perpendicular and parallel to the plane, the resulting equations are found to be quite different from those obtained in Example 1.

ΣF perpendicular to the plane:

$$N-100\cos 30°-P\sin 30°=0$$

ΣF parallel to the plane:

$$P\cos 30°-100\sin 30°-F'=0$$

The first equation shows that in this case the normal pressure N is not equal to the component of the weight, but is also influenced by the force P. The second equation shows that the force P must oppose the component of the weight that is parallel to the plane, in addition to the frictional resistance F'. A third equation $F'=f'N$ may be written. These three equations may be solved simultaneously for P, F', and N as required.

A much better method of solution for this problem is as follows: In Fig. 265 draw the vertical line V and the normal line N making an angle of 30° with V. Since motion of the free body up the plane is impending, it follows from statement (a), Art. 74,

that the frictional force F' acts down the plane, and R makes an angle ϕ' with N as shown. The reaction R is the vectorial sum of F' and N.

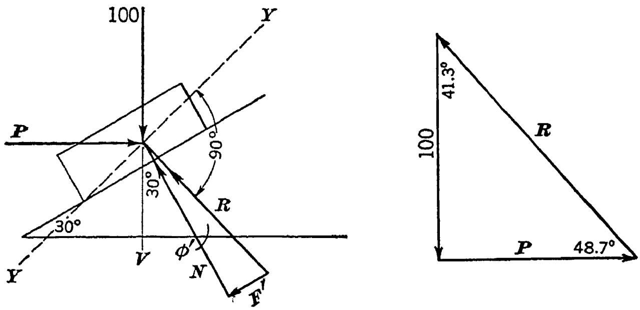


FIG. 265

The weight is held in equilibrium by the three forces P, 100 lb, and R, which is the resultant reaction of the plane on the weight. These three forces must pass through a common point, since they are maintaining equilibrium.

The forces P and R may now be found by any of the methods which have been developed for coplanar, concurrent force systems.

ΣF perpendicular to $R=0$:	ΣF perpendicular to $P=0$:
$P \cos 41.3° - 100 \cos 48.7° = 0$	$R \cos 41.3° - 100 = 0$
$P = 88.2$ lb	$R = 133.3$ lb

$$F' = 133.3 \sin \phi' = 26.3 \text{ lb}$$

The solution by the force triangle and sine law method is as follows:

$$\frac{P}{\sin 41.3°} = \frac{R}{\sin 90°} = \frac{100}{\sin 48.7°}$$

$$P = 88.2 \text{ lb and } R = 133.3 \text{ lb}$$

PROBLEMS

256. If a weight of 100 lb, resting on a horizontal plane, required a force of 30 lb to cause impending motion, what is the value of the coefficient of static friction? *Ans.* $f' = 0.3$.

257. Determine the coefficient of kinetic friction if a horizontal force of 40 lb will maintain uniform motion of a 100-lb block up a plane making an angle of 15° with the horizontal.

258. Solve for the force P, Fig. 266, which will just start the 100-lb block down the 15° plane. Take f' as 0.2.

259. If the elevator in Fig. 267 is to move up at a constant speed, and $f = 0.15$ at each of the four guides A, B, C, and D, what force P will be required? There is a workable clearance between the guides and the rails against which they rub.

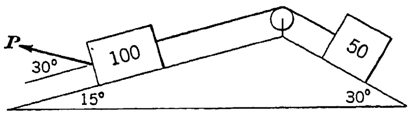


FIG. 266

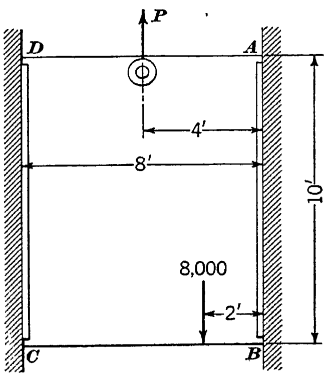


FIG. 267

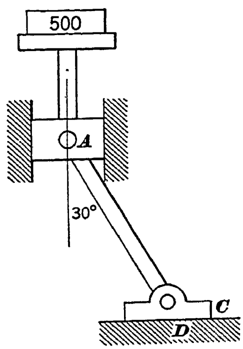


FIG. 268

260. Solve Problem 259 if the direction of motion is down.

261. A weight of 300 lb rests on a plane which makes an angle of 15° with the horizontal. What force P acting up the plane at 30° with the horizontal will cause impending motion? Take f' as 0.3.

262. What force acting horizontally on the weight of Problem 261 will cause motion down the plane to be impending?

263. The mechanism shown in Fig. 268 represents an unlubricated cross-head, A, supporting a 500-lb weight. Determine the compression in the connecting-rod AC, and the frictional force acting on the cross-head when downward motion of the 500-lb weight is impending. Take f' as 0.15. *Ans. 532 lb; 39.9 lb.*

264. If the conditions of Problem 263 were true, what would be the required value of the coefficient of friction for the surfaces shown at D?

76. Least Force.—It is sometimes desirable to know the least force which is necessary to cause or to prevent motion along a plane. The following example will illustrate the method of find-

ing the least force and also show that *the least force always has a definite direction, which is perpendicular to R.*

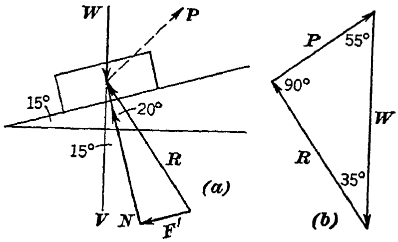


FIG. 269

EXAMPLE

Determine the amount and direction of the least force P which will cause motion of the weight W, Fig. 269 (a), to impend up the plane.

According to statement (a), Art. 74, the frictional resistance F' will oppose the motion of the free body W and will act down the plane. The resultant R will then act up to the left as shown.

This free body is in equilibrium because of the action of W, R, and some force P unknown in amount and direction. For equilibrium the vectors of these three forces must form a closed triangle. In Fig. 269 (b) vector W is drawn to any convenient scale. Through the lower end of W draw a line parallel to R. Through the other end of W draw the shortest line possible which will close the force triangle. This closing line determines the amount and the direction of the least force P.

Study of Fig. 269 (b) will show that the direction of the least force to cause or to prevent motion is always perpendicular to the reaction R of the plane.

PROBLEMS

265. A 500-lb weight rests on a plane which makes an angle of 30° with the horizontal. If $f = 0.3$, what are the amount and direction of the least force which will cause motion up the plane? *Ans. 364 lb; 46.7° with horizontal.*

266. If the kinetic coefficient of friction for the weight in Problem 265 is 0.2, what will be the amount and direction of the least force P which will cause uniform motion down the plane?

267. Determine the amount and direction of the least force which will prevent motion of the 200-lb weight up the 30° plane shown in Fig. 270. Assume that $f' = 0.3$.

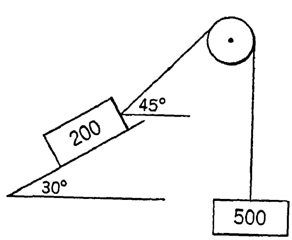


FIG. 270

77. Screw Friction.—A general treatment of screw friction would involve many different shapes of threads and cannot be con-

sidered in a book of this type. The discussion which follows applies only to screws with square threads, such as are found on jack-screws and some machine tools.

A square-threaded screw is essentially an inclined plane wound around a cylinder. This is clearly indicated in Fig. 271 (*b*) and 271 (*c*). If a horizontal force P is applied at the end of the lever shown in Fig. 271 (*a*), an equivalent horizontal force H will be developed at the mean radius of the thread. In this case,

$$P \times a = H \times r$$

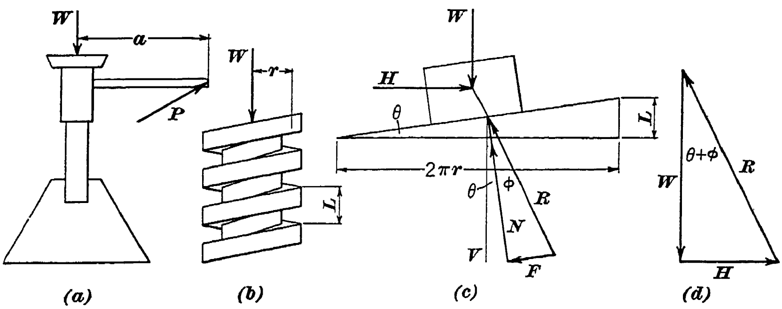


FIG. 271

The load carried by the screw acts vertically. The action of the screw is equivalent to pushing a block up an inclined plane as is indicated in Fig. 271 (*c*). The frictional force will oppose the motion of the block and act down the plane. If the rate of motion is constant, the block will be in equilibrium because of the action of the three forces H, W, and R. The force triangle for these forces is shown in Fig. 271 (*d*). In this triangle,

$$H = W \tan (\theta + \phi)$$

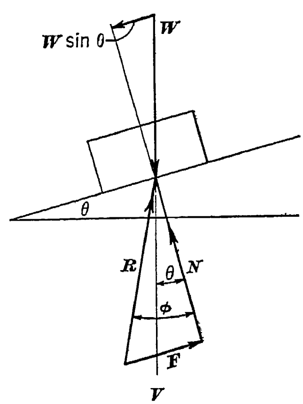


FIG. 272

Jack-screws must be self-locking to be useful. When the force P is removed from the lever, the screw must be able to support the weight without running down. When the force P is removed, the force H of Fig. 271 (*c*) becomes zero. The forces acting on the block are then reduced to two, W and R. If motion down the plane is impending,

the forces W and R must therefore be equal, opposite, and collinear, in order that the conditions of equilibrium may be satisfied.

If the screw is to be self-locking, R must fall to the left of V, as indicated in Fig. 272. The component of R that is parallel to the plane must be greater than $W \sin \theta$. For this to be true, ϕ must be greater than θ, or the slope of the screw thread must be less than the angle of friction.

PROBLEMS

268. The threads of a jack-screw have a mean radius of 2 in. and a pitch of $\frac{5}{8}$ in. If $f'=0.1$ and the lever has a length of 4 ft, what force must be applied at the end of the lever to just start a 10,000-lb weight upward? What force P will be necessary to lower the weight at a uniform rate if the kinetic coefficient f is 0.09? *Ans. 62.7 lb; 16.8 lb.*

269. Determine the maximum load that can be lifted at a uniform rate by a jack-screw with a mean thread diameter of 3 in. and a pitch of $\frac{3}{8}$ in., if a force of 50 lb is applied at the end of a 3-ft lever and $f=0.08$.

78. Wedge and Block.—In the design of various types of machine tools and other mechanisms, certain parts or units have sliding or plane frictional forces which act at two or more surfaces at the same time. The solution of this type of problem can best be illustrated by the action of the wedge and block.

EXAMPLE

Determine the force P which will cause the wedge shown in Fig. 273 to move to the right under the 1,000-lb block. The value of the coefficient is 0.364 for all surfaces.

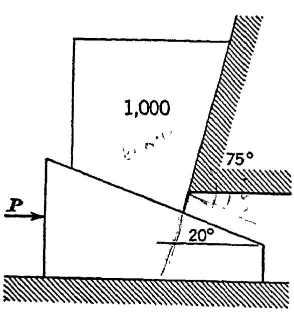


FIG. 273

The block is taken as the first free body, because it is acted upon by the known force of 1,000 lb. In Fig. 274 (a) draw the lines marked H and N_1, making an angle of 15° with each other. When the wedge moves to the right, the block will move up relative to the wall. The free body (block) is in motion relative to the wall; therefore, by statement (a), Art. 74, the frictional force parallel to the wall surface opposes the motion and acts downward. The force R_1, the resultant of F_1 and N_1, makes an angle of 20° with N_1 or is 5° above H.

Draw the lines V and N_2, making an angle of 20° with each other at the lower surface of the block. Consider the relative motion of the two surfaces in contact at this point. The free body (block) stands still while the wedge moves by. By statement (*b*), Art. 74, the frictional force F_2 acts in the direction of motion or down to the right. The force R_2, the resultant of F_2 and N_2, acts up to the right at an angle of 20° with N_2 and 40° with V.

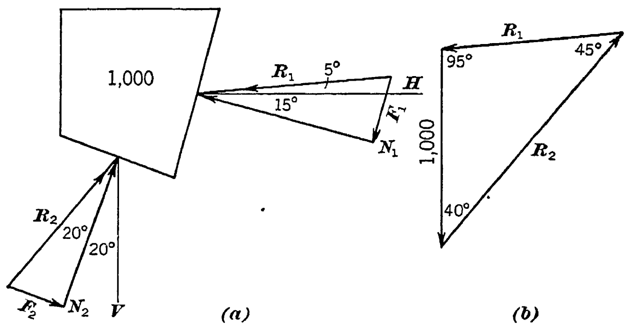


FIG. 274

The block is in equilibrium because of the action of the three forces, 1,000, R_1, and R_2. The force triangle for these forces is given in Fig. 274 (*b*). Thus,

$$\frac{R_1}{\sin 40°}=\frac{R_2}{\sin 95°}=\frac{1{,}000}{\sin 45°}$$

$$R_1=910 \text{ lb and } R_2=1{,}410 \text{ lb}$$

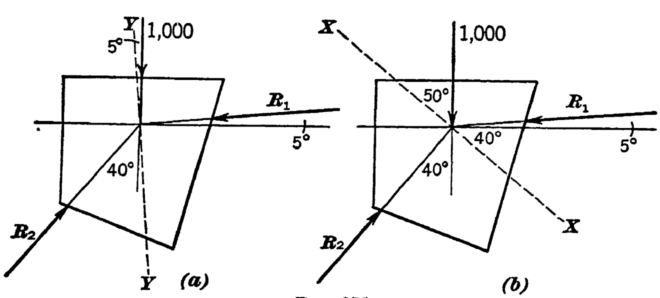


FIG. 275

The values of R_1 and R_2 may also be determined by the summation method. The free-body diagrams for this solution are given in Figs. 275 (*a*) and 275 (*b*).

ΣF perpendicular to $R_1=0$ $\qquad$ ΣF perpendicular to $R_2=0$

$R_2 \cos 45° - 1{,}000 \cos 5° = 0$ $\qquad$ $R_1 \cos 45° - 1{,}000 \cos 50° = 0$

$R_2 = 1{,}410$ lb $\qquad$ $R_1 = 910$ lb

In Fig. 276 (a), the wedge is used as the free body. At the upper surface draw V' and N_2' as shown. Remembering that the wedge is now the free body, consider the motion at the upper surface of the wedge. The wedge is in motion relative to the block. According to statement (a), friction opposes the motion. The force F_2' acts up to the left and R_2' acts down to the left at an angle of 20° with N_2' and 40° with V'. It will be observed that R_2 and R_2' are equal, opposite, and collinear—action and reaction.

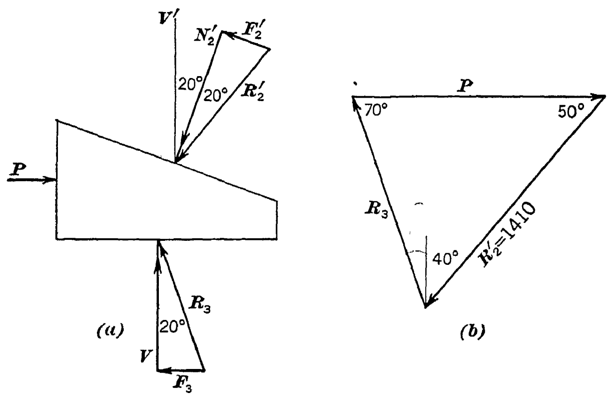


Fig. 276

At the lower surface of the wedge, the free body (wedge) is in motion relative to the horizontal plane. Statement (a) again applies, and R_3 acts up to the left at 20° with V. Fig. 276 (b) shows the force triangle formed from R_2' and the unknowns R_3 and P. In this triangle,

$$\frac{P}{\sin 60°} = \frac{R_3}{\sin 50°} = \frac{1{,}410}{\sin 70°}$$

$$P = 1{,}300 \text{ lb and } R_3 = 1{,}150 \text{ lb}$$

The student should check the above results by the summation method.

PROBLEMS

270. In the example just solved, determine the force P necessary to just start the wedge moving to the left. Take f' as 0.2126. *Ans. 67.3 lb.*

271. If $f' = 0.2679$, what force P is required to support the 2,000-lb load shown in Fig. 277?

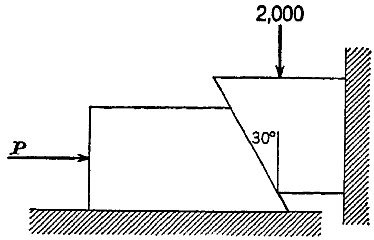


FIG. 277

272. In Problem 271 what would be the amount and direction of the least force P?

273. If in Fig. 277 the force P is removed, to what value must the 30° angle be changed in order that the device may be self-locking (impending downward motion caused by the 2,000-lb load)? Assume that $f' = 0.2679$.

274. Determine the values of forces P_1 and P_2 which will just prevent downward motion of the wedge carrying the 5,000-lb weight in Fig. 278. The value of f' for all surfaces is 0.3.

275. Compute the value of the force P necessary to force the wedge in Fig. 279 down if $\phi = 20°$. *Ans. 1,176 lb.*

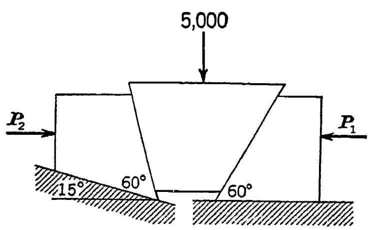


FIG. 278

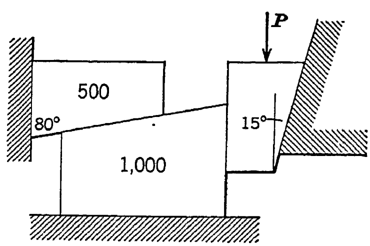


FIG. 279

79. Axle Friction and Friction Circle.—Fig. 280 (*a*) represents a pulley and shaft with an excessively loose bearing. The shaft is turning clockwise. If the bearing were frictionless, the reaction or supporting force of the bearing on the shaft would be normal to the surfaces at the point of contact and would pass through the center of the shaft. Because of friction between the surfaces of the shaft and bearing, the shaft—as it rolls—tends to climb up the surface of the bearing until equilibrium is reached between the climbing and slipping actions of the shaft. Such a point is reached at A where the shaft and bearing make contact over a very short arc or possibly along a line normal to the plane of the paper. A normal N to the surface of contact at A will pass

through the center of the shaft. The tangential frictional force F will act up to the right. The resultant R of N and F will pass through A at an angle ϕ with N. If a circle, called the friction circle, is now drawn with the center of the shaft as its center and with R as a tangent, the radius of the friction circle will be $r \sin \phi$, where r is the radius of the shaft.

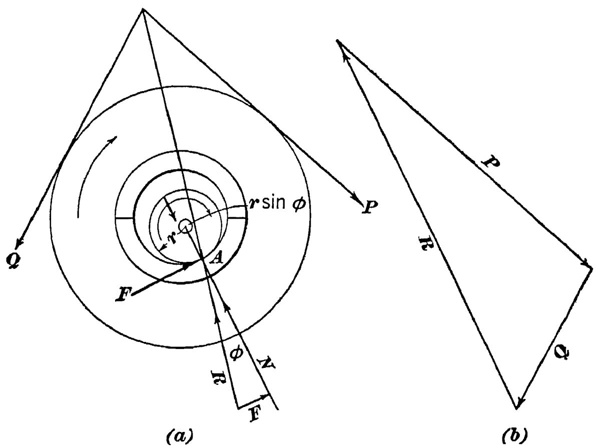


FIG. 280

If their weights are neglected, the pulley and the shaft are held in equilibrium by the three forces P, Q, and the resultant bearing reaction R. The reaction R must pass through the intersection of forces P and Q and must be tangent to the friction circle. Two such tangents can be drawn. The correct position of the tangent in any case can be determined by one of the following rules:

(*a*) Friction always decreases the lever arm of the driving force, and increases the lever arm of the driven force.

(*b*) The bearing reaction R is tangent to the friction circle on the side toward which the shaft rolls in the bearing.

(*c*) Place an arrow on either the journal or the bearing which will indicate the action (tension or compression) of that member on the other member. At right angles to this arrow place another arrow which indicates the direction of rotation of the first member relative to the second member. Continue the second arrow until it is parallel with the first and is pointing in the same direction. The reaction will be tangent to the friction circle on the side where the curved arrow ends.

These rules indicate that the bearing reaction is tangent to the friction circle on the side nearer the force P. A force triangle of the forces P, Q, and R may now be drawn, and R may be determined. This triangle is shown in Fig. 280 (*b*). The friction-circle method is convenient for graphical solutions.

When the angle ϕ is small ($\sin \phi = \tan \phi$), as is the case for well lubricated and properly fitted bearings, R and N are nearly equal, and F may be taken as equal to Rf without serious error.

The frictional force F may also be found by writing a moment equation with the axis of the shaft as the axis of moments. This equation is

$$(P-Q)d = Fr$$

EXAMPLE 1

Determine the amount of the frictional force and the coefficient of friction for uniform rotation of the pulley in Fig. 281. The pulley weighs 200 lb.

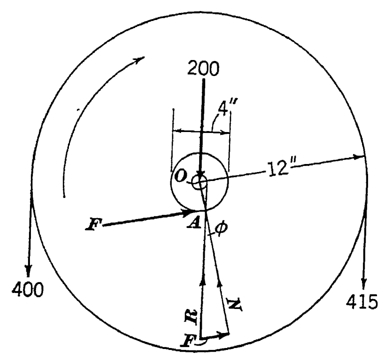


FIG. 281

Solution 1

$$\Sigma M_A = 0$$

Since the radius of the friction circle is $2f$,

$$415(12-2f) - 400(12+2f) - 200 \times 2f = 0$$
$$f = 0.08867$$
$$\Sigma V = 0$$

Since R is vertical,

$$R - 400 - 415 - 200 = 0$$
$$R = 1{,}015 \text{ lb}$$
$$F = R \sin \phi = R \tan \phi = Rf = 1{,}015 \times 0.08867 = 90 \text{ lb}$$

Solution 2

$$\Sigma M_O = 0$$
$$(415-400)12 - F\times 2 = 0$$
$$F = 90 \text{ lb}$$

$$\Sigma V = 0$$
$$R - 400 - 415 - 200 = 0$$
$$R = 1{,}015$$
$$F = R \sin\phi = R \tan\phi = Rf$$
$$90 = 1{,}015\, f$$
$$f = 0.08867$$

EXAMPLE 2

A loaded freight car weighs 300,000 lb. The wheels are 33 in. in diameter and the axles are 6 in. in diameter. Determine the force required to move the car if the coefficient of axle friction is 0.06.

Solution 1

In Fig. 282 (a) the entire weight of the car is assumed to be carried by one wheel, and P_b is the force parallel to the track which is required to move the car (overcome bearing friction). Fig. 282 (b) shows the wheel and axle as a free body which is held in equilibrium by the three forces P_b, R, and W. The force

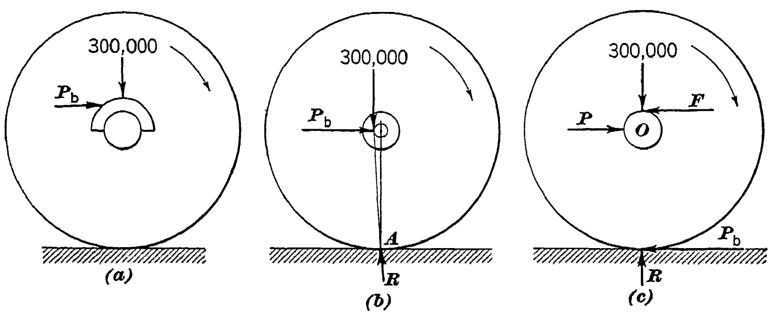


FIG. 282

R passes through the point of contact of the wheel and the track and is tangent to the friction circle on the left side because the axle tends to roll to the left on the bearing surface when the wheel turns clockwise. Forces P_b and W should intersect on the line of action of force R, but it is convenient to place P_b so that it

passes through the center of the shaft. The weight W will then be approximately tangent to the friction circle; and, if it is drawn tangent, as in Fig. 282 (b), the error is negligible.

The radius of the friction circle is $fr = 0.06 \times 3 = 0.018$ in.

$$\Sigma M_A = 0$$
$$P_b \times 16.5 - 300{,}000 \times 0.06 \times 3 = 0$$
$$P_b = 3{,}270 \text{ lb}$$

Solution 2

Fig. 282 (c) is another approximately accurate free-body diagram of the wheel. Here the assumption is made that the frictional force is due only to the pressure W of the weight on the axle. Since the force P also causes pressure of the bearing on the axle, the true value of F should be determined from the resultant of P and the 300,000-lb weight. However, the force P is so small compared to W that this resultant can be taken as W. In this case, P_b is the force exerted on the wheel by the track in causing the wheel to turn.

$$\Sigma M_O = 0$$
$$P_b \times 16.5 - 300{,}000 \times 0.06 \times 3 = 0$$
$$P_b = 3{,}270 \text{ lb}$$

PROBLEMS

276. Determine the amount of the force P necessary to raise the 500-lb weight shown in Fig. 283. The pulley and shaft weigh 200 lb and $f = 0.1$. *Ans. 307 lb.*

277. If a 3-in. diameter shaft, which carries a 24-in. diameter pulley with downward vertical belt pulls of 1,000 and 950 lb, turns at a uniform rate of speed, what frictional force is developed, and what is the coefficient of friction for the bearing? Explain the results.

278. A freight car weighs 200,000 lb. The car journals are 5 in. in diameter, and the wheels are 33 in. in diameter. If $f' = 0.08$, what force must be applied to the car to cause it to move? What is the greatest slope on which the car will stand? (Take a wheel as the free body. Assume R to be vertical.)

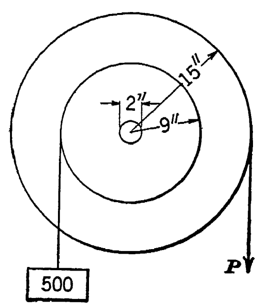


Fig. 283

279. Solve Problem 276 graphically, if the force P, Fig. 283, is acting at an angle of 30° with the horizontal. Neglect the weght of the pulley and the shaft.

280. Determine the force P in Fig. 283 that will allow the 500-lb weight to move downward at a uniform rate, if $f = 0.06$.

80. Rolling Resistance.—When a wheel or a cylinder rolls on a surface, either the wheel or the surface or both are deformed, the amount of the deformation of each material depending on the relative hardnesses of the materials.

In Fig. 284 (*a*), a wheel of hard material, such as steel, is rolling over a material sufficiently soft to permit us to assume that all the deformation takes place in the soft material. This soft material tends to pile up in front of the wheel, and the wheel acts as if it were climbing over a slight obstruction. The resultant reaction of the surface is represented by R.

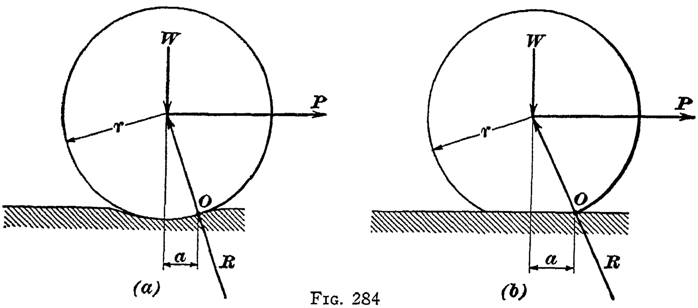


FIG. 284

In Fig. 284 (*b*) a wheel of soft material is rolling over a hard surface. In this case the assumption is that all the deformation occurs in the wheel. The reaction of the surface will pass through the center of the wheel and will lie in front of the normal radius, as shown. This case is somewhat similar to the action of rolling a rectangular block. In the case of a rigid block the reaction passes through the corner and the center of the block. Since the wheel is flexible, the corner is distorted or flattened, and the line of action of the resultant moves nearer to the normal radius.

For the average case of rolling, the conditions are in between the two which have just been considered. Both materials are more of less deformed. The reaction R will pass through the center of the wheel and will lie a distance a inches in front of the normal radius.

If a moment equation is written with the axis of moments on a line through O, the following equation will be obtained:

$$Pr = Wa$$

Experiments seem to show that the distance a is a constant for any given material, and that it is independent of the size or weight

of the body. There has been some disagreement on this matter, however. Some investigators maintain that a varies with the square root of the diameter. Because of this disagreement the values of a which are given below should be used with caution.

Elm on oak . . . . $a=0.0327$ in.
Steel on steel . . . . $a=0.007$ to 0.015 in.
Steel on wood . . . . $a=0.06$ to 0.10 in.

EXAMPLE

Find the force required to move a 150,000-lb car at a uniform rate on a level track. The car wheels are 33 in. in diameter, the axles are 5 in. in diameter, the coefficient of bearing friction is $f=0.04$, and the coefficient of rolling resistance is $a=0.015$ in.

Solution 1

Fig. 285 (*a*) is similar to Fig. 282 (*b*); but, because of the rolling resistance, the point A is 0.015 in. in front of the normal through the center of the wheel.

$$\Sigma M_A=0$$
$$16.5P-150{,}000(2.5\times 0.04+0.015)$$
$$P=1{,}045 \text{ lb}$$

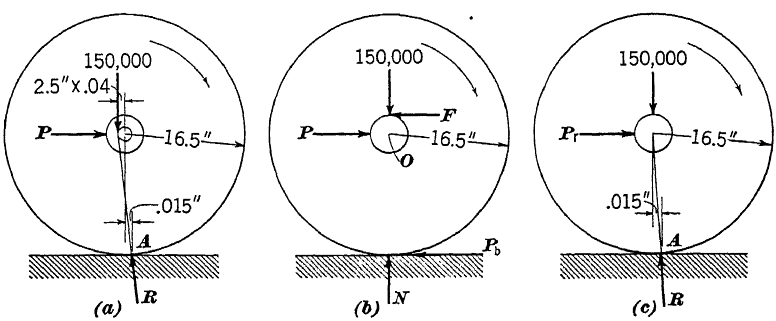


FIG. 285

Solution 2

Fig. 285 (*b*) is similar to Fig. 282 (*c*).

$$\Sigma M_O=0$$
$$P_b\times 16.5-150{,}000\times 0.04\times 2.5=0$$
$$P_b=909 \text{ lb}$$

In Fig. 285 (*c*),

$$\Sigma M_A = 0$$
$$P_r \times 16.5 - 150{,}000 \times 0.015 = 0$$
$$P_r = 136 \text{ lb}$$

$$P = 909 + 136 = 1{,}045 \text{ lb}$$

PROBLEMS

281. The coefficient of rolling resistance for a 33-in. car wheel is 0.02 in. If the wheel and its load weigh 10,000 lb, what horizontal force must be applied at the center of the axle to cause the wheel to move? *Ans. 12.1 lb.*

282. A freight car weighing 100,000 lb can be moved by a force of 1,300 lb applied parallel to the track. The axles are 5 in. in diameter, and $f = 0.08$ for axle bearings. Wheels are 33 in. in diameter. Determine the coefficient of rolling resistance.

283. A cast-iron machine, which weighs 5,000 lb, is to be moved on steel rollers 2 in. in diameter. The rollers roll on steel rails. What force is required to move the machine, if a for cast iron on steel is 0.012 in. and a for steel on steel is 0.008 in.?

284. If for the car of Problem 282 the coefficient of rolling resistance is 0.012 in., what total force will be required to move the car up a 5% grade?

81. Belt Friction.—The amount of power which can be transmitted by a belt or rope drive depends on the frictional force which can be developed at the surface of contact between the belt and pulley and on the speed of the belt.

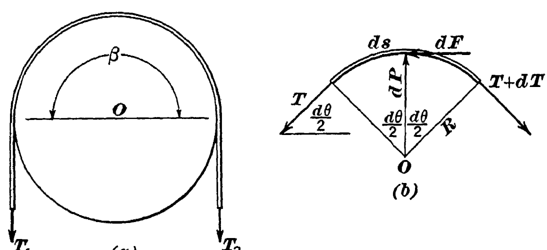


FIG. 286

In Fig. 286 (*a*), a belt is shown making contact with a pulley through an angle of β radians. The quantities T_2 and T_1 represent the tensions on the tight and loose sides of the belt, respectively. The pulley is being driven in a clockwise direction by the greater tension T_2.

A differential portion of the belt, of length ds, is shown as a free body in Fig. 286 (*b*). The normal pressure between the belt

and the pulley for this portion of the belt is dP; the frictional force developed at the surface of contact is dF; and dT is the difference between the tensions on the tight and loose sides of the belt.

When slipping of the belt on the pulley is impending, the free body is in equilibrium because of the action of the forces shown.

$$\Sigma F_v = 0$$

$$dP - T \sin \frac{d\theta}{2} - (T + dT) \sin \frac{d\theta}{2} = 0$$

$$dP - 2T \sin \frac{d\theta}{2} - dT \sin \frac{d\theta}{2} = 0$$

The term $\sin \frac{d\theta}{2}$ may be written as $\frac{d\theta}{2}$, since the sine of a small angle is equal to the angle in radians; moreover, the product of dT and $\frac{d\theta}{2}$ may be disregarded as it is a very small quantity. Thus,

$$dP = T\, d\theta \tag{1}$$

Also,

$$\Sigma M_O = 0$$
$$dF \times R - dT \times R = 0$$
$$dF = dT$$

When slipping of the belt is impending or occurring,

$$dF = dP \times f \text{ by definition}$$
$$dP \times f = dT$$
$$dP = \frac{dT}{f} \tag{2}$$

From equations (1) and (2),

$$T\, d\theta = \frac{dT}{f}$$

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\beta f\, d\theta$$

in which β must be expressed in radians.

$$\log_e \frac{T_2}{T_1} = f\, \beta$$

This equation may be changed to common logarithms by dividing by 2.31.

$$\log_{10}\frac{T_2}{T_1}=0.4343\,f\,\beta$$

The exponential form is

$$\frac{T_2}{T_1}=e^{f\beta}$$

In these equations T_2 *must always represent the larger of the two belt pulls,* and f is the coefficient of static friction. But, if the belt slips, the kinetic coefficient of friction must be used. It therefore follows that *the equations can be used only when slipping is impending or occurring.* The equations are also not to be used for high belt speeds because then the inertia effect of the belt reduces the pressure between the pulley and the belt.

EXAMPLE

If the pulley in Fig. 287 is turning clockwise, what force is acting on the pin A? Assume that $f=0.4$.

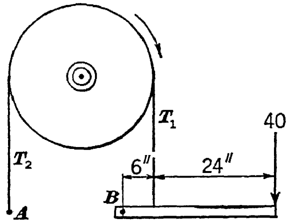


FIG. 287

Because of the frictional force between the belt and the pulley, the pull in the belt on the right side will be reduced and the pull on the pin A will be increased. The pull on pin A must therefore be represented by T_2, and the pull on the brake-lever by T_1.

Using the lever as the free body, solve for T_1 by taking moments with respect to an axis through B. Thus, T_1 is found to be 200 lb.

The angle of contact of the brake with the pulley is 180°, or $\beta=\pi$ radians.

$$\frac{T_2}{200}=2.718^{\pi\times 0.4}$$

$$T_2=200\times 2.718^{1.256}$$

$$T_2=702 \text{ lb}$$

PROBLEMS

285. A belt is in contact with a pulley through an angle of 150°. If the coefficient of static friction is 0.4 and the tension on the slack side of the belt is 100 lb, what is the maximum allowable pull on the tension side? *Ans. 285 lb.*

286. If a rope has two complete turns around a post, what force must be applied to the free end to support a 12,000-lb weight which is attached to the other end? Assume that $f' = 0.5$.

287. If a brake band is in contact with the brake pulley through an angle of 270° and the tensions at the ends of the band are 100 and 300 lb, what is the coefficient of friction?

288. If the brake of Problem 287 has a coefficient of 0.35 and the two tensions are 100 and 400 lb, what is the required angle of contact for the brake band?

289. A belt has an angle of contact of 120° with a pulley 5 ft in diameter. How many foot-pounds of torque will the belt exert on the pulley, if the tension on the tight side of the belt is 2,500 lb and $f' = \frac{1}{3}$?

82. Pivot, Ring Bearing, or Plate Clutch Friction.—In pivot or ring bearings and plate clutches, Fig. 288, there is relative motion or a tendency for relative motion of the plane surfaces of the annular rings. The wear of such rubbing surfaces is directly proportional to the radial distance ρ out from the center and the unit normal pressure p. Therefore, for uniform wear,

$$p\rho = \text{constant} = k$$

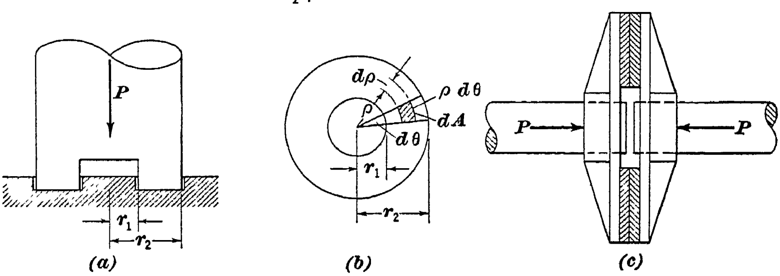


FIG. 288

If at first the unit pressure is uniform over the total area, the greatest wear will occur at the outside edge of the surface, since the relative motion is greatest there. Soon, as a result of the unequal wear, a readjustment of the unit pressures will be accomplished and a condition of uniform wear over the surface will be approached. This condition of uniform wear necessitates extremely high pressures over the central portion of the plates, if the relation $p\rho = k$ is to be maintained. This explains why the central portion of a pivot bearing is removed.

In Fig. 288 (*b*) the element of area $dA = \rho\, d\rho\, d\theta$ and $dP = p\, \rho\, d\rho\, d\theta = k\, d\rho\, d\theta$. The frictional force between the plates on area dA is $f\, dP = f\, k\, d\rho\, d\theta$, and the moment of this frictional force

about an axis through the center of the bearing is $dM = \rho f k \, d\rho \, d\theta$. Therefore, for the whole plate the resultant moment is

$$M = \int \rho f k \, d\rho \, d\theta = f k \int_{r_1}^{r_2} \rho \, d\rho \int_0^{2\pi} d\theta = f k \pi (r_2^2 - r_1^2) \qquad (1)$$

$$P = \int dP = \int k \, d\rho \, d\theta = k \int_{r_1}^{r_2} d\rho \int_0^{2\pi} d\theta = k \times 2\pi (r_2 - r_1) \qquad (2)$$

Eliminate k from equations (1) and (2). Then,

$$M = f P \frac{(r_2 + r_1)}{2}$$

The twisting moment transmitted is therefore equivalent to the product of $f P$ and the mean radius $\frac{r_2 + r_1}{2}$ of the bearing.

PROBLEMS

290. A plate clutch has two plates each having an outside diameter of 12 in. and an inside diameter of 7 in. What torque in foot-pounds will the clutch transmit if $f' = 0.35$ and the total normal pressure on the plates is 2,000 lb?

291. A propeller shaft 6 in. in diameter has an end thrust of 150,000 lb. If the outside diameter of each of eight collar bearings, which are attached to the shaft, is 12 in. and $f = 0.04$, what frictional torque is developed at the bearing? What is the average normal pressure on the bearing rings?

REVIEW PROBLEMS

292. A 100-lb block rests on a horizontal plane. If a 30-lb force acting to the right at an angle of 30° above the horizontal will cause impending motion, determine the frictional force, the angle of friction, and the coefficient of friction. *Ans. 25.98 lb; 17°; 0.3056.*

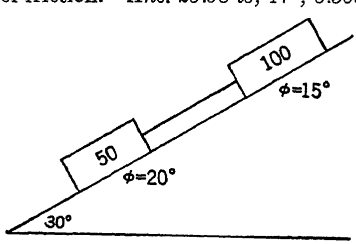


FIG. 289

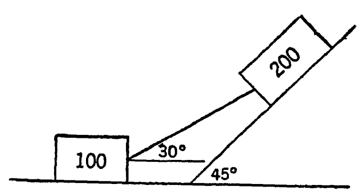


FIG. 290

293. Determine the least force which, when applied to the 100-lb block in Fig. 289, will prevent motion. What will be the least force which will cause motion up the plane to impend?

294. What is the value of the least force which can be applied to the 100-lb block in Fig. 290 to prevent motion, if $f' = 0.2$?

295. Derive an expression for the maximum value of W, in terms of a, b, and f, for the hanger shown in Fig. 291. Assume that $f' = 0.3$. Discuss this problem in terms of the theory explained in Art. 8, Chapter 1.

296. Determine the minimum value which the distance a in Fig. 291 may have, if $W = 100$ lb, $b = 6$ in., and $f' = 0.3$.

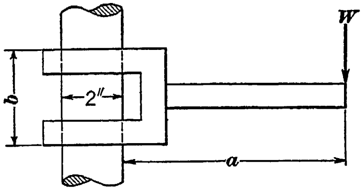


FIG. 291

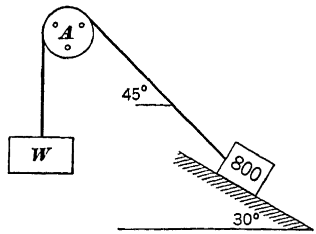


FIG. 292

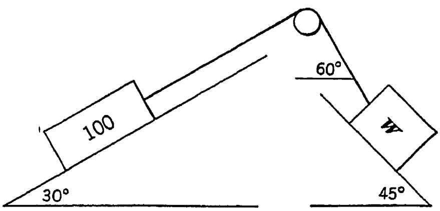


FIG. 293

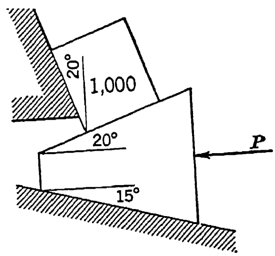


FIG. 294

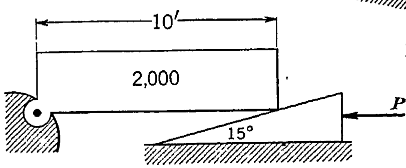


FIG. 295

297. A 20-ft ladder weighing 80 lb leans against a vertical wall at an angle of 30° with the wall. If $f' = 0.3$ for the wall and the floor, now far up the ladder can a 180-lb man go?

298. Find W, Fig. 292, for uniform motion of the 800-lb weight down the 30° plane. Assume that $f = 0.2$ for all surfaces and that the cylindrical surface A is fixed.

299. Calculate the value of W, Fig. 293, which will cause the 100-lb weight to start up the plane if $f' = 0.3$. *Ans. 136.5 lb.*

300. If W in Problem 299 is 300-lb, what force acting horizontally on the 100-lb block will cause motion down the 30° plane to impend?

301. What force must be applied at the end of a lever 2 ft long if a 6,000-lb weight is to be raised by a jack-screw? The mean thread diameter is 1.8 in. and the pitch is 0.4 in. The coefficient of friction for the screw thread is 0.15.

302. What is the value of the force P, Fig. 294, if the wedge is to move to the left and $f = 0.3$?

303. Calculate the value of force P, Fig. 295, which will start the wedge moving to the left if $f' = 0.2679$. *Ans. 847 lb.*

304. Compute the magnitude of the force P in Fig. 296 if motion to the left of the 500-lb block is impending and $\phi' = 12°$.

305. Determine the magnitude and direction of the least force P for impending motion of the 500-lb block of Fig. 296 to the left, if $f' = 0.2679$.

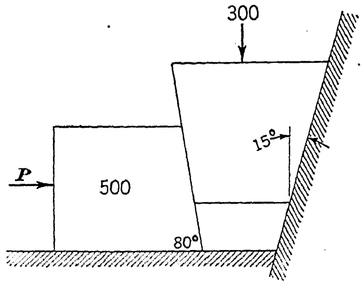


FIG. 296

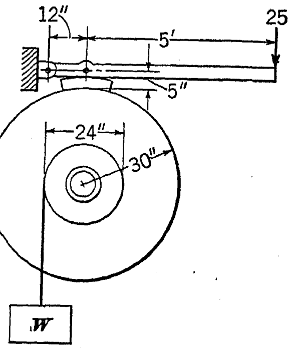


FIG. 300

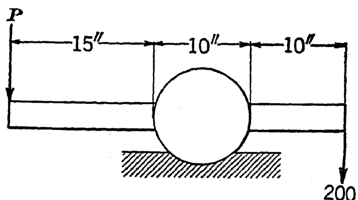


FIG. 297

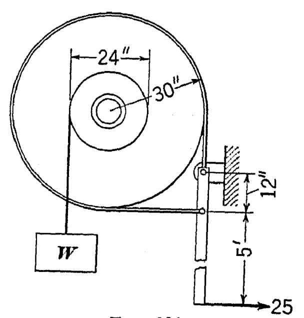


FIG. 301

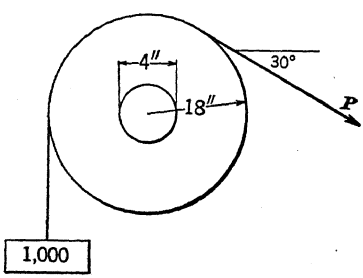


FIG. 298

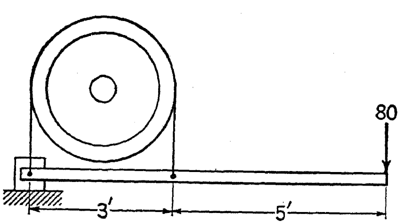


FIG. 299

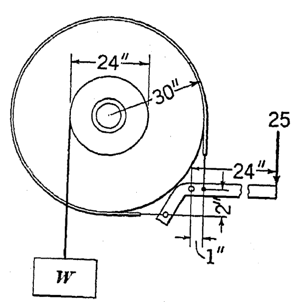


FIG. 302

306. Determine the force P, Fig. 297, required to cause motion in a counter-clockwise direction if $f' = 0.15$. What force will prevent motion in a clockwise direction?

307. Calculate the value of the force P, Fig. 298, which will just lift the 1,000-lb load. The coefficient of friction for the bearing is 0.15. What are the amount and direction of the bearing reaction?

308. If the pulley in Fig. 299 rotates clockwise, what is the torque of the frictional force? What is the torque if the rotation is in the other direction? Assume that $f = 0.4$. *Ans. 9,640 in.-lb; 2,750 in.-lb.*

309. Solve for the load W which the 25-lb force in Fig. 300 will just support, if $f' = 0.3$ for the brake shoe. The normal pressure of the brake shoe is assumed to be uniformly distributed over the area of contact.

310. Determine the load W which the 25-lb force in Fig. 301 will just support. The coefficient of friction for the band brake is $f' = 0.3$.

311. The coefficient of friction for the band brake in Fig. 302 is $f' = 0.3$. Determine the load which the 25-lb force will support.

312. Compare the relative efficiencies of the three brakes shown in Figs. 300, 301, and 302.

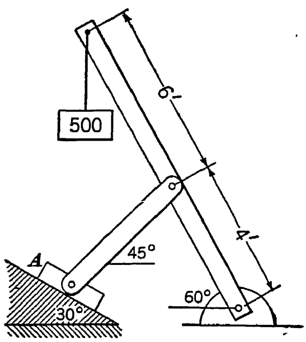


FIG. 303

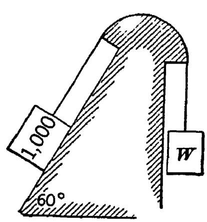


FIG. 304

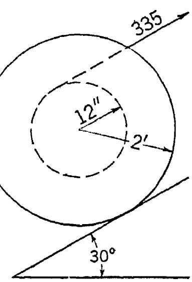


FIG. 305

313. What is the value of the coefficient of friction at the surface of block A, Fig. 303, if motion is impending?

314. If $f = 0.3$ for all surfaces in Fig. 304 and the 1,000-lb weight moves up the 60° plane at a uniform rate, what is the value of W?

315. Determine the force required to move a 100,000-lb freight car down a 2% grade at a uniform rate. The diameter of the wheels is 33 in., that of the axles is 5 in., $f = 0.09$, and $a = 0.02$ in.

316. A 200,000-lb car is being pulled up a 2% grade by a rope which is wound around a capstan. If the pull on the free end of the rope is 50 lb and $f' = 0.4$ for the capstan, how many turns are necessary? The diameter of the car wheels is 33 in., the axle diameter is 6 in., the coefficient of bearing friction is $f = 0.04$, and that for rolling resistance is $a = 0.015$ in.

317. Determine the coefficient of rolling resistance, a inches, if the 1,000-lb cable reel shown in Fig. 305 is moving at a constant speed.

318. The two plates of a clutch have an outside diameter of 24 in. and an inside diameter of 12 in.; and the coefficient of friction $f' = 0.4$. What torque can the plates transmit when there is a normal pressure of 80 psi between them?

319. A pivot bearing has an outside diameter of 12 in. and a counterbore 4 in. in diameter. If the normal pressure on the bearings is 4,000 lb and $f = 0.15$, what torque will be required to turn the pivot?

CHAPTER 10

CENTROIDS AND CENTERS OF GRAVITY

83. First Moments.—In the solutions of many problems in Mechanics and Strength of Materials, certain terms which involve the product of the length of a line, an area, a volume, or a mass and a distance from some point, line, or plane occur in the equations. These terms are known as the first moments of the lines, areas, volumes, or masses because of their similarity to terms representing the moment of a force with respect to an axis.

The present chapter and the two succeeding chapters on moment of inertia are devoted to the development of certain techniques, principles, and mathematical expressions which are constantly reoccurring in engineering calculations. The ability to use these techniques, principles, and expressions constitutes an essential tool of Mechanics.

84. Centroid and Center of Gravity Defined.—If a body is made up of a group of elementary masses of differential magnitudes, each of which is acted upon by a gravitational pull, there is a point in the mass through which the resultant of this system of essentially parallel forces will pass, regardless of how the mass may be rotated in relation to the surface of the earth. This statement is true only if the effect on the pull of gravity caused by the small changes in the distances of the various particles from the center of the earth is neglected. This discrepancy and that of the slightly non-parallel lines of force are unimportant in engineering calculations. The point in the body so described is the center of mass or center of gravity of the body.

The center of mass or center of gravity may also be defined as that point at which the entire mass of the body could be concentrated and still have the same moment with respect to any given axis as when the mass was in its original distributed state.

Another definition of center of mass or center of gravity is that it is a fixed point in the body on which the body will balance regardless of how the body is rotated.

The statements just made indicate that it is possible to locate the center of gravity of a body by experimental means. If a body

is suspended in two different positions by attaching a wire first at any given point and then at a second point, the intersection of the lines of action of the wire for the two points of suspension will be the center of gravity of the body.

The centroid of a volume and the center of gravity or center of mass of the volume may or may not be the same point.

The centroid of a volume and the center of gravity of a body which has the same size and shape and a constant density throughout are the same point. If some portions of the body have greater density than the others, the centroid and the center of gravity are not the same point.

The location of the centroid depends only on the geometrical form of the body. The position of the center of gravity is affected by a variation in the density of the different parts of the body. The centroid and the center of gravity or mass center of a solid homogeneous disk or cylinder is at the exact geometrical center of the disk. If the same disk were made of non-homogeneous material, the centroid of the volume would still be the geometrical center of the disk, but the location of the center of gravity would depend on the arrangement of the non-homogeneous material and might be at some distance from the centroid.

Lines and areas have no mass or volume and therefore have no center of gravity; but they do have centroids.

A line can be regarded as a homogeneous wire with an infinitely small cross-section. The center of gravity of the wire is then the centroid of the line. In a similar manner the centroid of an area can be described as the center of gravity of an infinitely thin plate.

85. Determination of the Centroid of an Area.—In Art. 16 the Principle of Moments was developed for a resultant force and its component forces. In Art. 26 this principle was used to determine the location of the resultant of a system of parallel forces.

The principle will now be restated in terms of areas instead of forces. *The moment of an area, with respect to a line or plane, is equal to the algebraic sum of the moments of its component areas, with respect to the same line or plane.*

This principle may be used to locate the centroid of any plane area by determining the distance to the centroid from each of two intersecting axes which lie in the plane of the area.

Fig. 306 represents any irregular plane area A. The small areas dA_1, dA_2, dA_3, ... are differential areas, the sum of which

is equal to the total area A. If the Principle of Moments is applied to the area A and its component areas, the following equations are obtained:

$$\Sigma M_y = A\,\bar{x}$$
$$A\,\bar{x} = x_1\,dA_1 + x_2\,dA_2 + x_3\,dA_3 + \ldots\ldots$$

In this equation, $\bar{x}$ (read *x bar*) is the distance to the centroid from the Y axis. Also,

$$\Sigma M_x = A\,\bar{y}$$
$$A\,\bar{y} = y_1\,dA_1 + y_2\,dA_2 + y_3\,dA_3 + \ldots\ldots$$

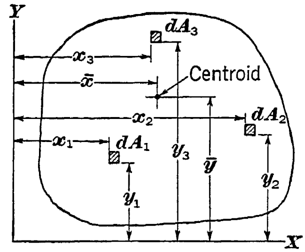


FIG. 306

where $\bar{y}$ is the distance to the centroid from the X axis. The centroid is thus definitely located.

When locating a centroid, the proper selection of axes is very important. It is always desirable *to select the axes in such a manner that the entire area will lie in a single quadrant,* as shown in Fig. 306, or to draw the axes tangent to the external boundary of the figure. If the axes are so placed, confusion of signs will be avoided. There will be no negative signs unless a portion of the figure is considered removed, as in the case of a hollow square or any other figure which is obtained by cutting away a portion of the original figure.

The equations just given may be more conveniently written as follows:

$$A\,\bar{x} = \int x\,dA \qquad\qquad A\,\bar{y} = \int y\,dA$$

86. Centroids of Lines, Surfaces, Volumes, and Masses.—In Art. 85 the general equations for locating the centroid of an area were developed. Similar equations may be written for locating the centroids of any lines, surfaces, volumes, and masses. The various equations are:

$$L\,\bar{x} = \int x\,dL \qquad L\,\bar{y} = \int y\,dL \qquad L\,\bar{z} = \int z\,dL$$
$$S\,\bar{x} = \int x\,dS \qquad S\,\bar{y} = \int y\,dS \qquad S\,\bar{z} = \int z\,dS$$
$$V\,\bar{x} = \int x\,dV \qquad V\,\bar{y} = \int y\,dV \qquad V\,\bar{z} = \int z\,dV$$
$$M\,\bar{x} = \int x\,dM \qquad M\,\bar{y} = \int y\,dM \qquad M\,\bar{z} = \int z\,dM$$
$$W\,\bar{x} = \int x\,dW \qquad W\,\bar{y} = \int y\,dW \qquad W\,\bar{z} = \int z\,dW$$

87. **Symmetry.**—The task of locating centroids can very often be much simplified by careful selection of axes. In many cases there are lines or planes of symmetry which assist in locating the centroids. For any figure which has a line or plane of symmetry, that line or plane always contains the centroid of the figure.

If a plane area has two lines of symmetry, the intersection of the two lines is the centroid. If a solid has two planes of symmetry, the intersection of the two planes is an axis of symmetry and contains the centroid. If there are three planes of symmetry, those planes will intersect in a common point and that point is the centroid of the figure. A line or plane of symmetry always contains the centroid.

The centroid of a straight line is at its mid-point.

The centroid of a circular arc is on the bisecting radius.

The centroid of a cone or a pyramid is on the line of intersection of any two planes of symmetry, or on the axis of symmetry.

The centroid of a hemisphere is also on the line of intersection of any two planes of symmetry, or on the axis of symmetry.

88. **Rules for the Proper Selection of the Element for Integration.**—A large number of the problems which occur in engineering calculations, and which involve finding of centroids, require little use of the calculus for their solution. Many of these cases can be broken up into simple geometric forms for each of which the location of the centroid is known.

In those cases which require the use of the calculus for their solution, the differential element should be selected according to one of the two rules which follow:

(*a*) The element should be so selected that all parts of the element are the same distance from the axis or plane with respect to which moments are being written.

(*b*) If the centroid of the element is known, the expression may consist of the product of the element and the distance to the centroid of the element from the axis or plane with respect to which moments are being written.

89. **Centroids of Elementary Forms by Integration.**—The centroids of several of the more elementary forms will now be located. The student is advised to study the procedure carefully and to learn the location of the centroid of the figure used in each of the examples which follow.

EXAMPLE 1

Centroid of a circular arc. Because of the symmetry of the figure, the centroid is on the bisecting radius. This radius is taken as the X axis, as indicated in Fig. 307. Then $\bar{y}=0$. If $\bar{x}$ is determined, the centroid will be definitely located. The general form of the equation for this case is

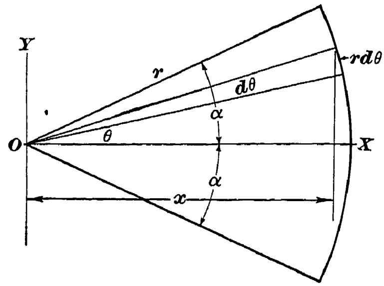


FIG. 307

$$L\,\bar{x}=\int x\,dL$$

Since $L=2\,r\,\alpha$, $x=r\cos\theta$, and $dL=r\,d\theta$,

$$2\,r\,\alpha\,\bar{x}=\int_{-\alpha}^{\alpha} r\cos\theta\,r\,d\theta=2\,r^2\sin\alpha$$

$$\bar{x}=\frac{2\,r^2\sin\alpha}{2\,r\,\alpha}=\frac{r\sin\alpha}{\alpha}$$

For a semicircular arc, $\alpha=\frac{\pi}{2}$ radians and $\bar{x}=\frac{2\,r}{\pi}$.

EXAMPLE 2

Centroid of a sector of a circle.—If the X axis is selected so that it bisects the sector, as in Fig. 308, $\bar{y}=0$ and the centroid will be located when $\bar{x}$ is determined.

$$A\,\bar{x}=\int x\,dA$$

Since $A=r^2\,\alpha$, $x=\rho\cos\theta$, and $dA=d\rho\,\rho\,d\theta$,

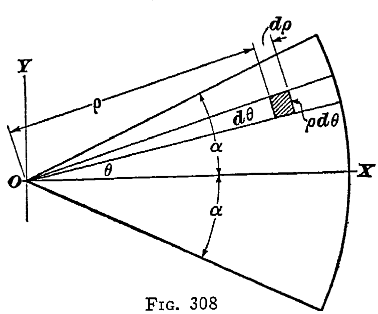


FIG. 308

$$r^2\,\alpha\,\bar{x}=\int_0^r\int_{-\alpha}^{\alpha}\rho\cos\theta\,d\rho\,\rho\,d\theta$$

$$r^2\,\alpha\,\bar{x}=\frac{r^3}{3}\,2\sin\alpha$$

$$\bar{x}=\frac{2\,r\sin\alpha}{3\,\alpha}$$

When the sector is a semicircle, $\alpha=\frac{\pi}{2}$ radians and $\bar{x}=\frac{4\,r}{3\,\pi}$.

If the sector is a quadrant, the perpendicular distance to the centroid from either bounding radius is also $\frac{4\,r}{3\,\pi}$.

EXAMPLE 3

Centroid of a triangle.—According to the laws of symmetry, the centroid must be on the line connecting any vertex with the center of the opposite side, which may be taken as the base. From Fig. 309 the following equations are obtained:

$$A\,\bar{y}=\int y\,dA$$

Since $A=\frac{1}{2}\,b\,h$, $dA=u\,dy$, and $\frac{u}{b}=\frac{y}{h}$ or $u=\frac{b\,y}{h}$,

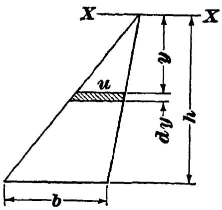


Fig. 309

$$\frac{1}{2}\,b\,h\,\bar{y}=\int_0^h y\,u\,dy$$

$$\frac{1}{2}\,b\,h\,\bar{y}=\frac{b}{h}\int_0^h y^2\,dy=\frac{b\,h^3}{3\,h}$$

$$\bar{y}=\frac{2}{3}\,h$$

EXAMPLE 4

Centroid of a cone or pyramid.—By symmetry the centroid of the cone or pyramid is located on the line connecting the vertex with the center of the base. In Fig. 310 this line of symmetry is taken as the X axis.

In this case, $A=$area of base; $a=$area of selected section; $V=\frac{A\,h}{3}$; $dV=a\,dx$. For a right circular cone,

$$\frac{a}{A}=\frac{\pi\,y^2}{\pi\,R^2}$$

From similar triangles, $\frac{y}{R}=\frac{x}{h}$. Hence,

$$\frac{a}{A}=\frac{x^2}{h^2}$$

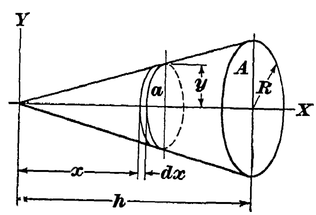


Fig. 310

This relationship for $\frac{a}{A}$ will be true for any shape of section.

$$V\,\bar{x} = \int x\,dV = \int x\,a\,dx$$

$$\frac{A\,h}{3}\,\bar{x} = \int_0^h x\,\frac{A\,x^2}{h^2}\,dx = \frac{A\,h^4}{4\,h^2}$$

$$\bar{x} = \frac{3}{4}\,h$$

EXAMPLE 5

Centroid of a hemisphere.—In Fig. 311 the axis of symmetry was selected as the Y axis.

$$V\,\bar{y} = \int y\,dV$$

$$\frac{2}{3}\,\pi\,R^3\,\bar{y} = \int_0^R y\,\pi\,x^2\,dy$$

Since $x^2 = R^2 - y^2$,

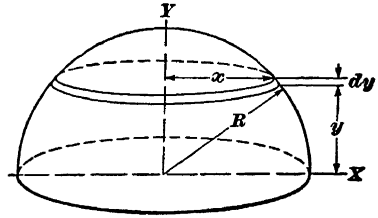


FIG. 311

$$\frac{2}{3}\,\pi\,R^3\,\bar{y} = \pi \int_0^R y\,(R^2 - y^2)\,dy$$

$$\frac{2}{3}\,\pi\,R^3\,\bar{y} = \frac{\pi\,R^4}{4}$$

$$\bar{y} = \frac{3}{8}\,R$$

PROBLEMS

320. Locate the centroid of the area in Fig. 312 between the parabola, the X axis, and the line $x = a$. *Ans.* $\frac{3}{5}\,a$; $\frac{3}{8}\,b$.

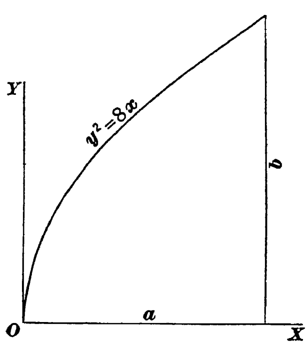


FIG. 312

321. Locate the centroid of the area in Fig. 312 between the parabola, the Y axis, and the line $y = b$.

322. Show that the centroid of a hemispherical surface with radius R is at a distance $\frac{R}{2}$ from the diametrical plane.

323. Solve Example 2 by applying rule (*b*), Art. 88.

324. Prove by integration that the centroid of a quadrant of radius R is $\frac{4R}{3\pi}$ distant from the bounding radius. Use a sector as the element of area.

325. Solve Problem 324 by using a strip parallel to the bounding radius as the element of area. The equation of the circle is $x^2+y^2=R^2$.

326. Determine the distance to the centroid of the curved surface of a cone from the vertex.

90. Centroids and Centers of Gravity of Composite Figures. The centroid of a figure which is made up of several parts may be easily found, if the centroids of the component parts of the figure are known.

EXAMPLE 1

Locate the centroid of the plane area shown in Fig. 313.

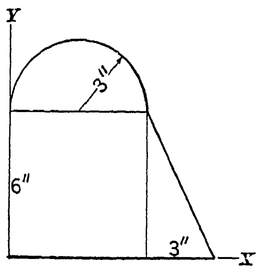


Fig. 313

The X and Y axes are selected so that the entire area is in the first quadrant. By the Principle of Moments, the moment of the resultant area with respect to the X axis is equal to the algebraic sum of the moments of the component areas with respect to the same axis. In a similar manner an equation can be written for the Y axis.

The centroid of the semicircular area is located on the bisecting radius at distance $\frac{4R}{3\pi}$ from the diameter, as given in Art. 89, Example 2.

Resultant area $A=6\times6+\frac{6\times3}{2}+\frac{\pi\,3^2}{2}=59.1$ sq in.

$$\Sigma M_x = A\,\bar{y}$$

$$59.1\,\bar{y}=36\times3+9\times2+14.1\left(\frac{4\times3}{3\pi}+6\right)$$

$$\bar{y}=3.86 \text{ in.}$$

$$\Sigma M_y = A\,\bar{x}$$

$$59.1\,\bar{x}=36\times3+9\times7+14.1\times3$$

$$\bar{x}=3.61 \text{ in.}$$

EXAMPLE 2

Locate the center of gravity of the body shown in Fig. 314. The cone is of steel and the hemisphere is of lead. Steel weighs 490 lb per cu ft and lead weighs 710 lb.

$$\text{Weight} = \frac{\pi\, 3^2}{3} \times 6 \times \frac{490}{1{,}728} + \frac{2}{3}\,\pi\, 3^3 \times \frac{710}{1{,}728}$$

$$= 16.0 + 23.2 = 39.2 \text{ lb}$$

$$\Sigma M_x = \int y\, dW$$

$$39.2\,\bar{y} = 16.0 \times \frac{3}{4} \times 6 + 23.2 \left(6 + \frac{3}{8} \times 3\right)$$

$$\bar{y} = 6.05 \text{ in.}$$

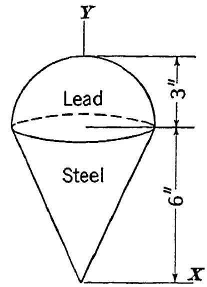


FIG. 314

PROBLEMS

327. Locate the centroid of the area shown in Fig. 315. *Ans. 4.17 in.; 5.67 in.*

328. Determine the value of $\bar{y}$ in Fig. 316.

329. A hollow hemisphere has an outside radius of 6 in. and an inside radius of 5 in. Locate the centroid. (HINT: $V\bar{y} = V_1\bar{y}_1 - V_2\bar{y}_2$)

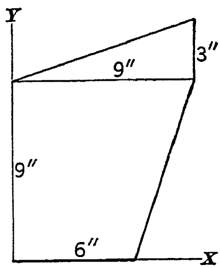


FIG. 315

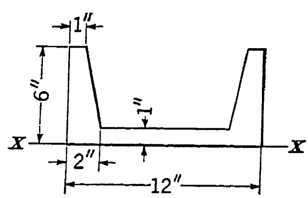


FIG. 316

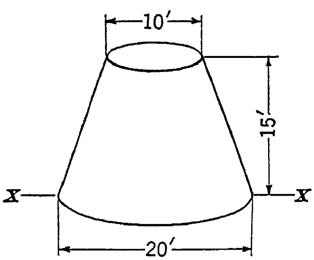


FIG. 317

330. Locate the centroid of the frustum of a cone, which is shown in Fig. 317.

331. If in Problem 329 the hollow hemisphere is of steel and the hollow part is filled with lead, where is the center of gravity? Specific weights of steel and lead are, respectively, 490 and 710 lb per cu ft.

332. Locate the centroid of a wire bent into the form of the external boundary line in Fig. 313.

333. Determine the position of the centroid of the body shown in Fig. 318

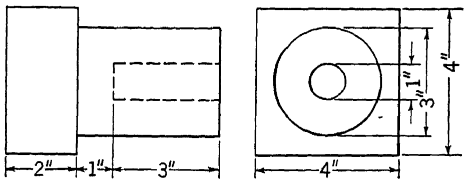


FIG. 318

91. Theorems of Pappus and Guldinus.—1. If any plane curve is revolved about any axis (in the plane of the curve) which does not pass through the curve, the area generated is equal to the product of the length of the curve and the length of the arc generated by the centroid of the curve.

It should be noted here that the curve may touch the axis about which it is being rotated but *cannot pass through the axis.*

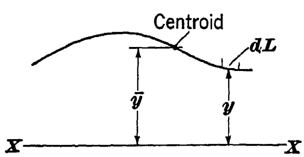


FIG. 319

If L is the length of the curve shown in Fig. 319 and $\bar{y}$ is the distance from the centroid of the curve to the axis about which rotation is taking place, then

$$\Sigma M_x = L\,\bar{y}$$

$$L\,\bar{y} = \int y\,dL \qquad (1)$$

$$S = 2\,\pi \int y\,dL \qquad (2)$$

Substituting from (1) for $\int y\,dL$ gives

$$S = 2\,\pi\,\bar{y}\,L$$

2. If any plane area is revolved about any axis in its plane which does not pass through the area, the volume which is generated is equal to the product of the area and the length of the arc generated by the centroid of the area.

The axis may touch the boundary of the area but *must not pass through the area.*

If A is the area of the triangle shown in Fig. 320 and $\bar{y}$ is the distance to the centroid of the triangle from the X axis, then

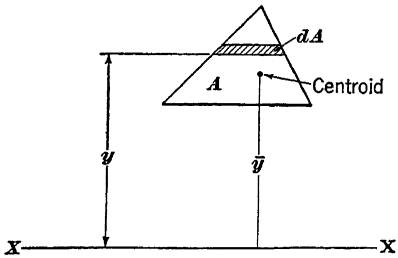


FIG. 320

$$\Sigma M_x = A\,\bar{y}$$
$$A\,\bar{y} = \int y\,dA \qquad (1)$$
$$V = 2\,\pi \int y\,dA \qquad (2)$$

Substituting from (1) for $\int y\,dA$ gives

$$V = 2\,\pi\,\bar{y}\,A$$

PROBLEMS

334. Determine the area of the curved surface of a cone, if the slant height is L and the radius of the base is R. *Ans.* $\pi\,R\,L$.

335. Compute the volume of a cone of altitude H and radius of base R.

336. Determine the volume of a sphere.

337. Show that the centroid of a semicircle is at a distance $\frac{4R}{3\pi}$ from the diameter.

338. Determine the area of the surface and the volume of the solid generated when a semicircular area is revolved about the X axis shown in Fig. 321. *Ans. 500.2 sq in.; 467 cu in.*

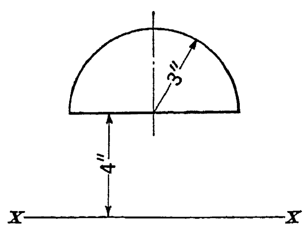


FIG. 321

339. Prove by the first theorem that the surface area of a sphere is $4\pi r^2$.

340. Prove that, if the area shown in Fig. 312 is revolved about the X axis, the volume generated will be half that of a cylinder with a radius b and an altitude a.

REVIEW PROBLEMS

341. Locate the centroid of the area shown in Fig. 322. *Ans. 8.19 in.; 4.07 in.*

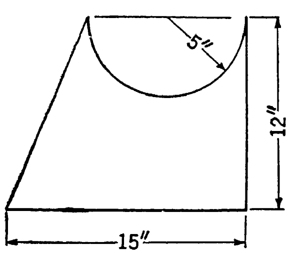


FIG. 322

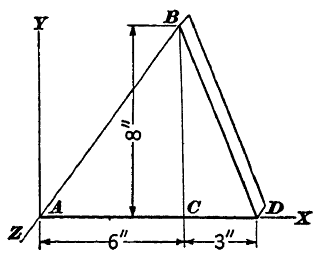


FIG. 323

342. The steel plate shown in Fig. 323, which is 2 in. thick, is cut and bent along the 8-in. dimension so that ACD becomes a 90° angle. Locate the center of gravity of the plate. Assume that the plate is almost cut through on the 8-in. line before the bending takes place.

343. Locate the centroid of the area which is shown in Fig. 324.

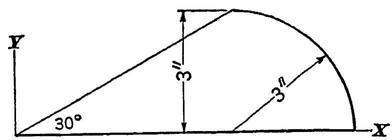


FIG. 324

344. Fig. 325 represents a cross-section through a concrete retaining wall and the earth fill behind the wall. Determine the distance from A to the line along which the resultant vertical load due to the weight of the wall and earth fill will act. Concrete weighs 150 lb per cu ft and earth weighs 100 lb per cu ft.

345. The pressure on a submerged surface is directly proportional to the distance below the surface of the water. What is the total pressure on a vertical gate 10 ft wide and 5 ft deep, if the upper edge of the gate is 10 ft below the surface of the water? Where does the resultant pressure act? *Ans. 12.67 ft.*

346. Derive a general formula for center of pressure and one for total pressure on a submerged surface. *Ans.* $y_{cp}=\frac{I}{\bar{y}A}$; $P=w\bar{y}A$.

347. Locate the center of gravity of the crankshaft shown in Fig. 326.

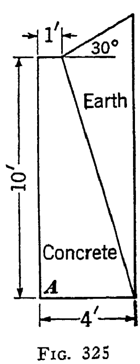


FIG. 325

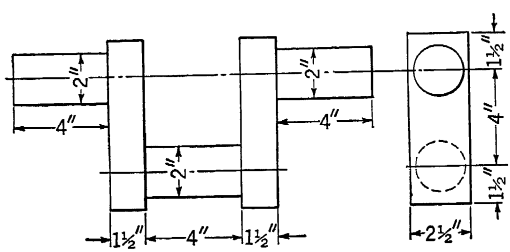


FIG. 326

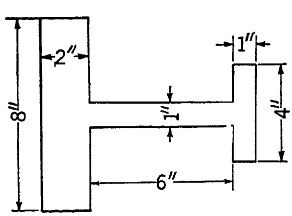


FIG. 327

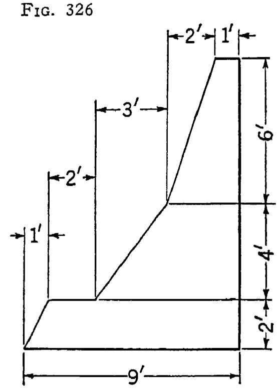


FIG. 328

348. Locate the centroid of the section shown in Fig. 327.

349. Compute the distances to the center of gravity of the wall shown in Fig. 328 from the vertical surface and the base. *Ans. 2.71 ft; 3.97 ft.*

350. Locate the center of gravity of the cylindrical solid shown in Fig. 329. Weight of cast iron is 450 lb per cu ft; of lead, 710 lb per cu ft.

351. Determine the location of the centroid of the shaded area shown in Fig. 330.

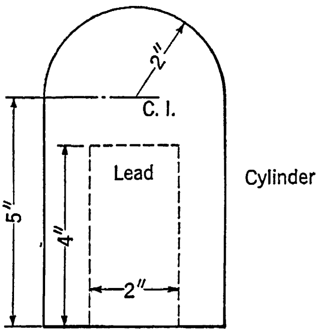


FIG. 329

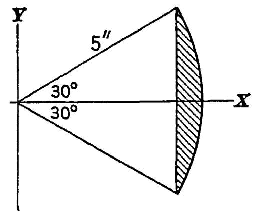


FIG. 330

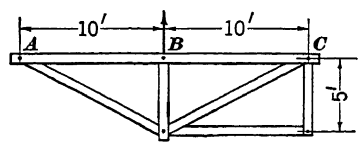


FIG. 332

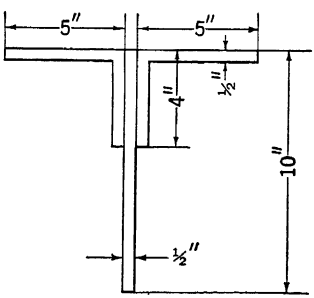


FIG. 331

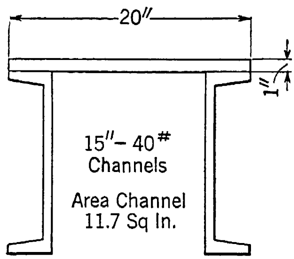


FIG. 333

352. Locate the centroid of the beam section shown in Fig. 331.

353. The frame shown in Fig. 332 is to be lifted by means of a rope which is attached at point *B*. Determine the slope of the line *AC* while the truss is being lifted. All members of the truss weigh 10 lb per ft. *Ans. 37.3°.*

354. ·Locate the centroid of the section which is shown in Fig. 333.

355. In Problem 354 substitute two 15″ 42.9-lb I-beams for the channels. Area of one I-beam is 12.49 sq in. Locate the centroia of the section.

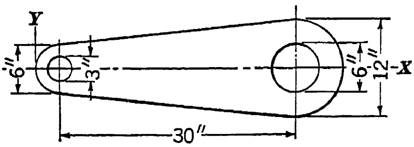


FIG. 334

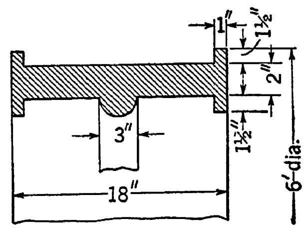


FIG. 335

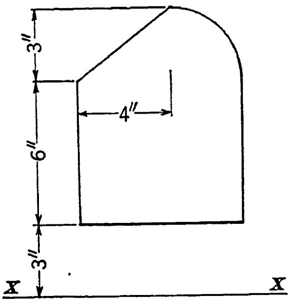


FIG. 336

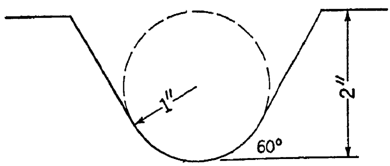


FIG. 337

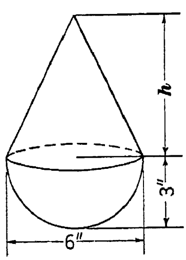


FIG. 338

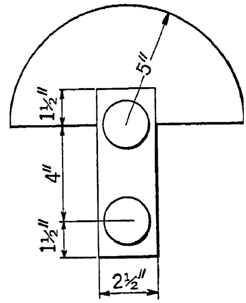


FIG. 339

356. Determine the value of $\bar{x}$ for the plate shown in Fig. 334.

357. The rim of a rope-drive wheel has a cross-section as shown in Fig. 335. If the wheel is made of cast iron, what does the rim weigh? The weight of cast iron is 450 lb per cu ft.

358. A wire bent in the form shown in Fig. 336 is revolved about the X axis. Determine the area of the surface generated.

359. If a notch of the form shown in Fig. 337 is cut around a steel shaft 12 in. in diameter, how much material will be removed? Steel weighs 490 lb per cu ft. *Ans. 41.59 lb.*

360. The equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Determine the distances to the centroid of one quadrant from the axes of the ellipse.

361. Determine the volume and the position of the centroid of the solid of revolution generated when one quadrant of the ellipse of Problem 360 is revolved about the X axis.

362. Determine the allowable height h of the cone mounted on a hemispherical base, as in Fig. 338, if the object is to be in stable equilibrium (always return to the position shown).

363. Two balance weights are to be added to the shaft of Problem 347. The weights are to be placed as shown in Fig. 339 and have a radius of 5 in. How thick should the weights be?

CHAPTER 11

SECOND MOMENTS OF AREA—MOMENTS OF INERTIA

92. **General Discussion.**—In the development of the fundamental equations of Strength of Materials, expressions of the form $\int x^2\,dA$, $\int y^2\,dA$, and $\int \rho^2\,dA$ are frequently encountered. These expressions occur in the formulas for bending of beams, deflection of beams, and twisting of shafts, and in the formulas for columns.

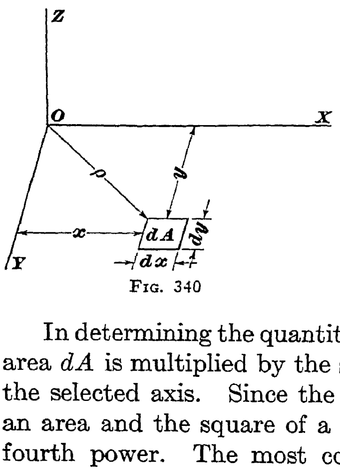


FIG. 340

The term dA represents an element of area of differential magnitude so arranged that all parts of the element are at the same distance from the axis of reference. The axis may be any axis in the plane of the area, such as either of the rectangular X and Y axes in Fig. 340 or an axis OZ perpendicular to the plane of the area and at a distance ρ from the area dA.

In determining the quantity known as the moment of inertia, the area dA is multiplied by the square of the distance x, y, or ρ from the selected axis. Since the moment of inertia is the product of an area and the square of a distance, it is a length raised to the fourth power. The most commonly used units are in.^{4} Since the distance is squared, it always has the positive sign. The area is inherently positive; therefore, *moment of inertia is always a positive quantity*. The various expressions for "Moment of Inertia of an Area," which are constantly occurring in certain engineering calculations, are simply a group of mathematical symbols that have no physical significance. Moment of inertia cannot be represented by a diagram or picture. For simplification of the mathematics, symbols such as I_x, I_y, and J have been substituted for the mathematical expressions $\int y^2\,dA$, $\int x^2\,dA$ and $\int \rho^2\,dA$, respectively.

These expressions were given the name Moment of Inertia because of their similarity to terms which occur in the study of rotating bodies. Inertia is a property of matter. Areas do not

possess this property; and the term "Moment of Inertia of an Area" is somewhat misleading. The name "Second Moment of Area" would be much more appropriate when applied to areas. However, custom of long standing demands that these terms be called Moment of Inertia of Area.

The moments of inertia with respect to the X and Y axes are indicated by the following expressions:

$$I_x = \int y^2\, dA \qquad\qquad I_y = \int x^2\, dA$$

The polar moment of inertia, or the moment of inertia with respect to an axis perpendicular to the plane of the area, is indicated by the expression

$$J = \int \rho^2\, dA$$

93. Radius of Gyration.—It is sometimes convenient to express the moment of inertia in the following manner:

$$I = Ak^2$$

from which

$$k^2 = \frac{I}{A}$$

The term k is known as the radius of gyration. It appears in many column formulas. The quantity k^2 is the mean or average value of the term x^2, y^2, or ρ^2 in the expression for moment of inertia. If the entire area could be concentrated into a single element of area, this area would be at a distance k from the axis of reference.

94. Moments of Inertia of Certain Fundamental Areas by Integration.—As in the work on centroids, it is necessary that the student know how to obtain the moment of inertia of each of certain common figures which occur frequently in problems in design.

In the formulas of strength of materials where the moment of inertia occurs, *the element of area always represents a strip of area, all parts of which are at the same distance from the axis of reference.* When selecting an element of area, it is necessary that this requirement be satisfied.[1]

Either single or multiple integration may be used.

---

[1] It is possible to compute the moment of inertia of an area without taking the element of area parallel to the axis, but the method given here is generally preferable. The other methods involve the use of the transfer theorem, Art. 95, or the application of special formulas.

EXAMPLE 1

Determine the moment of inertia and radius of gyration of a rectangle of base b and altitude h with respect to an axis through the centroid parallel to the base.

The rectangle is shown in Fig. 341.

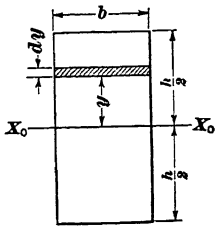


FIG. 341

$$I_{x_0}=\int y^2\,dA$$

$$I_{x_0}=\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\,b\,dy$$

$$I_{x_0}=\frac{1}{12}\,b\,h^3$$

$$k_{x_0}=\sqrt{\frac{b\,h^3}{12\,b\,h}}=\frac{h}{2\sqrt{3}}$$

EXAMPLE 2

Determine the moment of inertia and the radius of gyration of a triangle of base b and altitude h with respect to its base X_1X_1, Fig. 342, and also with respect to an axis through the centroid parallel to the base, as axis X_0X_0, Fig. 343.

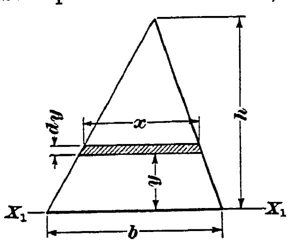


FIG. 342

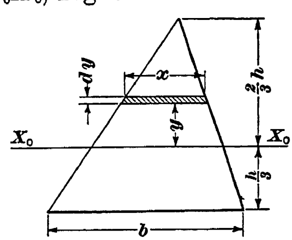


FIG. 343

$$I_b=\int y^2\,dA \text{ and } dA=x\,dy$$

$$I_b=\int y^2\,x\,dy;\ \frac{x}{b}=\frac{h-y}{h};\ x=\frac{b}{h}\,(h-y)$$

$$I_b=\frac{b}{h}\int_0^h y^2\,(h-y)\,dy$$

$$I_b=\frac{1}{12}\,b\,h^3 \text{ and } k_b=\sqrt{\frac{2\,b\,h^3}{12\,b\,h}}=\frac{h}{\sqrt{6}}$$

$$I_{x_0}=\int y^2\,dA \text{ and, in Fig. 343, } dA=x\,dy$$

$$I_{x_0}=\int_{-\frac{1}{3}h}^{\frac{2}{3}h} y^2\,x\,dy;\ \frac{x}{b}=\frac{\frac{2}{3}h-y}{h};\ x=\frac{2}{3}b-\frac{b}{h}y$$

$$I_{x_0}=\int_{-\frac{1}{3}h}^{\frac{2}{3}h} y^2\left(\frac{2}{3}b-\frac{b}{h}y\right)dy$$

$$I_{x_0}=\frac{2}{3}b\int_{-\frac{1}{3}h}^{\frac{2}{3}h} y^2\,dy-\frac{b}{h}\int_{-\frac{1}{3}h}^{\frac{2}{3}h} y^3\,dy$$

$$I_{x_0}=\frac{1}{36}b\,h^3 \text{ and } k_{x_0}=\sqrt{\frac{2\,b\,h^3}{36\,b\,h}}=\frac{h}{3\sqrt{2}}$$

EXAMPLE 3

Derive a formula for the moment of inertia of a circle of radius r with respect to a diameter. What is its radius of gyration?

One quarter of the circle is shown in Fig. 344.

$$I_x=\int y^2\,dA \text{ and } dA=\rho\,d\theta\,d\rho$$

$$y=\rho\sin\theta$$

$$I_x=\int\int \rho^2\sin^2\theta\,\rho\,d\theta\,d\rho$$

$$I_x=4\int_0^{\frac{\pi}{2}}\int_0^r \rho^3\,d\rho\,\sin^2\theta\,d\theta$$

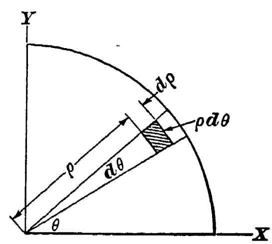


FIG. 344

$$I_x=r^4\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\,d\theta=r^4\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$I_x=\frac{\pi\,r^4}{4}$$

$$k_x=\sqrt{\frac{I}{A}}=\sqrt{\frac{\pi\,r^4}{4\pi\,r^2}}=\frac{r}{2}$$

EXAMPLE 4

What are the polar moment of inertia and radius of gyration of a circle of radius r with respect to an axis through its center, as indicated in Fig. 345?

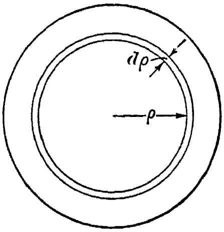


Fig. 345

$$J=\int \rho^2\,dA \text{ and } dA=2\pi\,\rho\,d\rho$$

$$J=2\pi\int_0^r \rho^3\,d\rho$$

$$J=\frac{1}{2}\pi\,r^4 \text{ and } k=\sqrt{\frac{\pi\,r^4}{2\,\pi\,r^2}}=\frac{r}{\sqrt{2}}$$

PROBLEMS

364. Determine the moment of inertia of the rectangle in Fig. 341 of base b and altitude h with respect to its base. *Ans.* $\frac{bh^3}{3}$.

365. Determine the moment of inertia of the triangle in Fig. 342 of base b and altitude h with respect to an axis through the vertex parallel to the base.

366. Determine the moment of inertia of the area inclosed by the parabola $y^2=4x$, the line $x=4$, and the X axis with respect to the X axis.

367. Using the element of area indicated in Fig. 346, determine the moment of inertia of the circle with respect to the X axis.

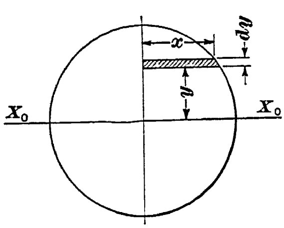


Fig. 346

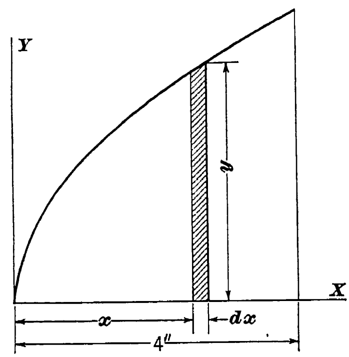


Fig. 347

368. Determine the polar moment of inertia of a rectangle with sides a and b with respect to an axis through the centroid of the rectangle.

369. Check the solution of Problem 366 by each of the two methods suggested in the footnote on page 191. In these solutions, use the element of area indicated in Fig. 347.

Suggestion: The element may be treated as a rectangle, for which $I_x=\frac{b\,h^3}{3}$ and $I_{x_0}\doteq\frac{b\,h^3}{12}$. By the first expression,

$$I_x=\int\frac{b\,h^3}{3}=\int_0^4\frac{y^3\,dx}{3}$$

By use of the second expression and the transfer formula,

$$I_x=\int\left[\frac{y^3\,dx}{12}+y\,dx\left(\frac{y}{2}\right)^2\right]$$

95. Transfer Formula for Parallel Axes.—In Fig. 348, A represents any plane area with the centroid on the X_0 axis and X_1 represents any other parallel axis.

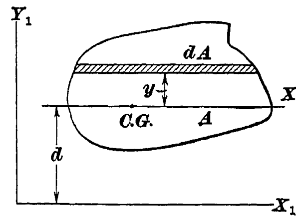


FIG. 348

$$I_{x_1} = \int (y+d)^2\, dA$$

$$I_{x_1} = \int y^2\, dA + 2d\int y\, dA + d^2 \int dA$$

$$\int y^2\, dA = I_{x_0} \text{ and } \int y\, dA = \bar{y}\, A$$

But $\bar{y}=0$, because X_0 is the centroidal axis. Hence,

$$I_{x_1} = I_{x_0} + A\, d^2$$

The student should note that this theorem may be applied only when *one of the two parallel lines is a centroid axis of the area A which is being transferred.* Examination of the equation also indicates that the moment of inertia with respect to the centroidal axis of the area A will always be smaller than the moment of inertia with respect to any other parallel axis.

If the transfer formula is divided by A, the following relationship is obtained:

$$k^2_{x_1} = k^2_{x_0} + d^2$$

EXAMPLE

Using the relation $I_{x_1} = \dfrac{b\, h^3}{12}$ for the moment of inertia of a triangle with respect to its base, determine the moment of inertia with respect to a parallel axis through the vertex of the triangle.

The conditions are represented in Fig. 349.

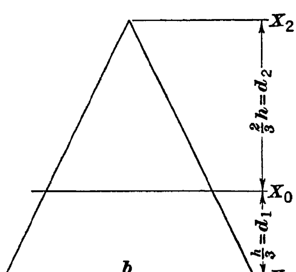


FIG. 349

$$I_{x_1} = I_{x_0} + A\, d_1^2$$

$$\frac{b\, h^3}{12} = I_{x_0} + \frac{b\, h}{2}\left(\frac{h}{3}\right)^2$$

$$I_{x_0} = \frac{b\, h^3}{36}$$

$$I_{x_2} = I_{x_0} + A\, d_2^2$$

$$I_{x_2} = \frac{b\, h^3}{36} + \frac{b\, h}{2}\left(\frac{2}{3}\, h\right)^2$$

$$I_{x_2} = \frac{b\, h^3}{4}$$

The two transfers can be combined into one equation, as follows:

$$I_{x_2} = I_{x_1} - A\,d_1^2 + A\,d_2^2$$

In transferring to the centroidal axis, the quantity Ad^2 is subtracted; and, in transferring from the centroidal axis, the quantity Ad^2 is added.

$$I_{x_2} = \frac{b\,h^3}{12} - \frac{b\,h}{2}\left(\frac{h}{3}\right)^2 + \frac{b\,h}{2}\left(\frac{2}{3}h\right)^2$$

$$I_{x_2} = \frac{b\,h^3}{4}$$

PROBLEMS

370. Given $I = \frac{\pi r^4}{4}$ for a circle of radius r with respect to a diameter, find I with respect to a tangent. *Ans.* $\frac{5}{4}\pi r^4$.

371. Given $I = \frac{bh^3}{4}$ with respect to an axis through the vertex parallel to the base of a triangle, determine the moment of inertia with respect to the base.

372. The moment of inertia of a circle with respect to a diameter is $\frac{\pi r^4}{4}$. Determine the moment of inertia of a semicircle with respect to a tangent parallel to the bounding diameter.

373. If I for a circle with respect to a diameter is $\frac{\pi r^4}{4}$, what is the moment of inertia of a quadrant of a circle with respect to a line through its centroid parallel to the limiting radius?

374. Given $\frac{bh^3}{12}$ for the moment of inertia of a triangle with respect to the base, find I about a line at a distance $\frac{h}{4}$ above the base and parallel to it.

96. Relation Between Rectangular and Polar Moments of Inertia. The moment of inertia of any area with respect to an axis normal to the area is known as the Polar Moment of Inertia of the area. It is designated by the letter J. Let dA represent any differential element of the plane area A shown in Fig. 350. Then,

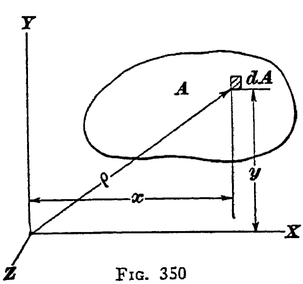


FIG. 350

$$I_z = J = \int \rho^2\, dA = \int (x^2+y^2)\, dA$$
$$J = \int x^2\, dA + \int y^2\, dA = I_x + I_y$$

The moment of inertia of an area with respect to any axis perpendicular to the area, or the polar moment of inertia of the area, is equal to the sum of the moments of inertia with respect to any two rectangular axes which intersect in the axis perpendicular to the plane.

PROBLEMS

375. Given a rectangle of sides a and b, determine J with respect to an axis through one corner of the rectangle. *Ans.* $\dfrac{ab(a^2+b^2)}{3}$.

376. Given an isosceles triangle of base b and altitude h, what is J with respect to an axis through the vertex perpendicular to the plane of the triangle?

377. Show that J for a circle is $\dfrac{\pi r^4}{2}$.

378. Determine J for a semicircular area of radius r with respect to an axis through the point of intersection of the bisecting radius and the circumference.

379. Write the expression for the polar moment of inertia of an ellipse with respect to an axis normal to the plane of the ellipse and passing through the positive end of the major axis of the ellipse. Use the relation $I_{x_0} = \dfrac{\pi ab^3}{4}$.

380. Prove that the moment of inertia of a square with respect to any axis drawn through the centroid of the square is a constant, $I = \dfrac{a^4}{12}$.

97. Transfer Formula for Polar Moment of Inertia.—Let X and Y, Fig. 351, be any axes at distances a and b from the centroidal axes X_0 and Y_0 of the area A.

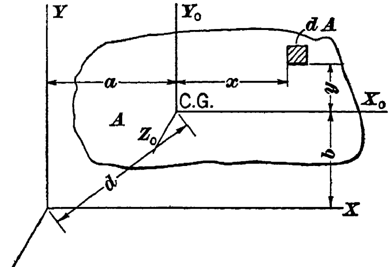


FIG. 351

$$I_z = J = \int [(a+x)^2 + (b+y)^2] dA$$
$$J = a^2 \int dA + 2a \int x\, dA + \int x^2\, dA + b^2 \int dA + 2b \int y\, dA + \int y^2\, dA$$

Since $\int x\,dA=\bar{x}\,A=0$ and $\int y\,dA=\bar{y}\,A=0$,

$$J=a^2\,A+I_{y_0}+b^2\,A+I_{x_0}$$

But $I_{x_0}+I_{y_0}=J_0$ and $a^2+b^2=d^2$. Hence,

$$J=J_0+A\,d^2$$

It is important to note that this equation has the same limitations as the transfer formula for rectangular moments of inertia. *One of the two parallel lines must pass through the centroid of the area A, which is being transferred.*

PROBLEMS

381. Determine the polar moment of inertia for a square with respect to an axis through a point half way between the center and one corner.

382. Compute J for an axis through the centroid of a quadrant of a circle.

383. What is the polar moment of inertia of a section through a hollow shaft, 6 in. in outside diameter and 3 in. in inside diameter, with respect to an axis 1 in. from the center?

98. Moments of Inertia of Composite Areas.—In general, when an area can be divided into component areas, such as rectangles, triangles, circles, or parts of a circle, the moment of inertia about any axis may be determined by one of the following methods.

1. Divide the area into its component areas. Write the moment of inertia of each of the component areas with respect to an axis through its own centroid, and parallel to the line with respect to which the moment of inertia of the entire figure is desired. By applying the transfer formula, the moment of inertia of the selected component area may then be obtained with respect to the desired axis. Repeat the process for each of the component areas. Add the results for the resultant moment of inertia of the entire figure.

2. In many cases it is possible to divide the area into component areas in such a manner that the moment of inertia of the entire area may be written with respect to a line which is parallel to the desired axis. By proper use of the transfer formula, the moment of inertia with respect to any desired parallel axis may be obtained. See Art. 95 for use of transfer formulas.

Where an area contains holes, or has parts of the area removed, such a part is treated similarly to any other component area, but the moment of inertia of the part is given the negative sign, and

is thus deducted from the composite moment of inertia of the whole area.

The application of the two methods will now be illustrated by examples.

EXAMPLE 1

Determine the moment of inertia of the area shown in Fig. 352 with respect to the X_1 axis.

By the first method.—The moment of inertia of each component area with respect to an axis through its own centroid parallel to the required X_1 axis is found first, as follows:

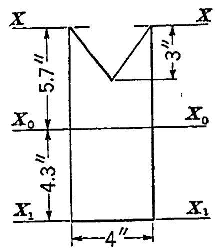


FIG. 352

Rectangle........ $\frac{bh^3}{12}=\frac{4\times 10^3}{12}=333.3$

Triangle.......... $\frac{bh^3}{36}=\frac{4\times 3^3}{36}=3$

Now transfer from the centroidal axis of each component area to the desired X_1 axis.

$$I_{x_1}=I_{x_0}+Ad^2$$

Rectangle..... $I_{x_1}=333.3+40\times 5^2=1{,}333.3$

Triangle...... $I_{x_1}=3+6\times 9^2=489$

For the composite area,

$$I_{x_1}=1{,}333.3-489=844.3 \text{ in.}^4$$

By the second method:

$$A=4\times 10-\frac{4\times 3}{2}=34$$

$$\Sigma M_x=A\,\bar{y}$$

$$34\,\bar{y}=40\times 5-6\times 1$$

$$\bar{y}=5.705 \text{ in.}$$

Write an expression for the moment of inertia for the entire area about the X axis.

$$I_x=\frac{4\times 10^3}{3}-\frac{4\times 3^3}{12}=1{,}333.3-9=1{,}324.3$$

Transfer to the X_0 axis.

$$I_{x_0}=I_x-A\,d^2$$

$$I_{x_0}=1{,}324.3-34\times 5.705^2=217.51$$

Transfer from the X_0 axis to the X_1 axis.

$$I_{x_1}=217.51+34\times 4.295^2=844.8 \text{ in.}^4$$

EXAMPLE 2

Compute I_{x_0} for the T section shown in Fig. 353.

Apply the second method:

$$\Sigma M_x = A\,\bar{y}$$

$$8\,\bar{y} = 4\times5\times2.5 - 3\times4\times2$$

$$\bar{y} = 3.25 \text{ in.}$$

$$I_x = \frac{4\times5^3}{3} - \frac{3\times4^3}{3} = 102.6$$

$$I_{x_0} = 102.6 - 8\times3.25^2 = 18.12 \text{ in.}^4$$

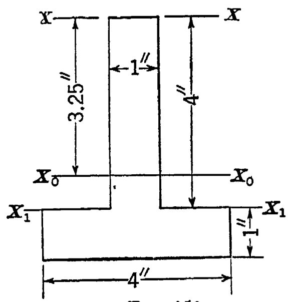


FIG. 353

The student should study these illustrative Examples carefully, observing that—when the transfer formula is applied—*one of the parallel axes is always a centroidal axis of the area A which is being transferred.*

PROBLEMS

384. Calculate I_{x_0} and I_{y_0} for Fig. 354. *Ans. 30.8 in.⁴; 10.8 in.⁴*

385. Compute the centroidal moments of inertia for Fig. 355.

386. Solve for I_{x_0} and I_{y_0} for the area shown in Fig. 356.

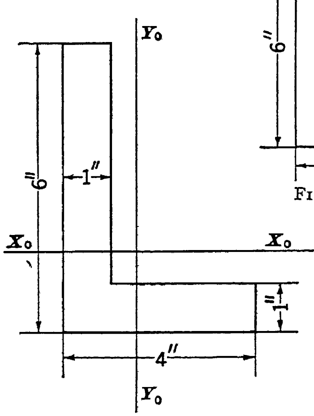


FIG. 354

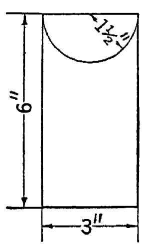


FIG. 355

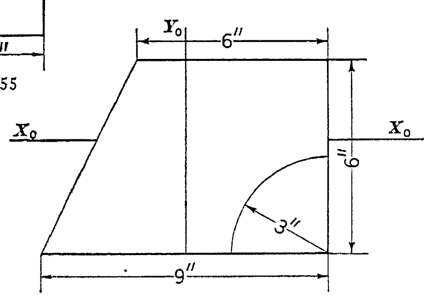


FIG. 356

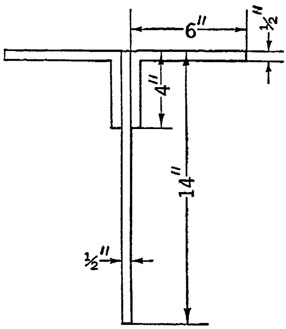


FIG. 357

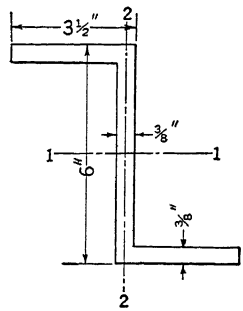


FIG. 358

387. Determine the centroidal moments of inertia for the beam section shown in Fig. 357.

388. A hollow circular column has an outside diameter of 12 in. and an inside diameter of 8 in. Determine the moment of inertia of the section with respect to a diameter, and also the radius of gyration. *Ans.* *817 in.*^{4}*; 3.6 in.*

389. Calculate I_{1-1} and I_{2-2} for the Z bar shown in Fig. 358.

390. Solve for I_{x_0} and I_{y_0} for the channel shown in Fig. 359.

391. Solve for I_{x_0} and I_{x_1} for the area shown in Fig. 360.

392. Determine the polar moment of inertia for the area in Fig. 360 with respect to an axis through the centroid.

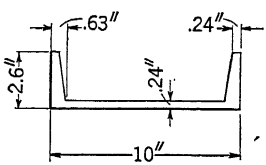


FIG. 359

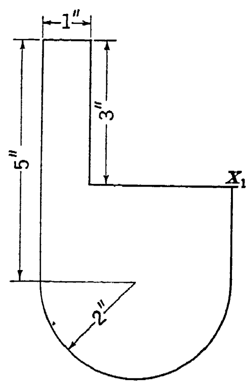


FIG. 360

99. Moment of Inertia by Approximate Method.—If an area is of such form that it cannot be divided into parts whose moments of inertia can be easily computed, or if it is difficult to set up an equation which will represent the boundary curve, the following approximate method may be used for determining the moment of inertia.

Let Fig. 361 represent any area of irregular shape whose moment of inertia about the X axis is desired. Divide the area into narrow strips parallel to the axis. The narrower the strips are made, the more accurate the results will be. Each of the narrow strips may be considered to be a rectangle. Then the moment of inertia of the area with respect to the X axis will be given by the following equation:

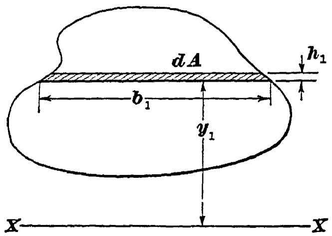


FIG. 361

$$I_x = \frac{b_1 h_1^3}{12} + b_1 h_1 y_1^2 + \frac{b_2 h_2^3}{12} + b_2 h_2 y_2^2 + \ldots\ldots$$

If the dimension h of each strip is made very small, the terms $\frac{b_1 h_1^3}{12}$ may be omitted without serious error. The equation then becomes

$$I_x = b_1 h_1 y_1^2 + b_2 h_2 y_2^2 + b_3 h_3 y_3^2 + \ldots\ldots$$

PROBLEMS

393. Determine the moment of inertia of a circle with 8-in. radius with respect to a diameter. Determine the percentage of error which this method produces.

394. Calculate I for a $3'' \times 4''$ rectangle with respect to the 3-in. side. Check the answer.

100. Products of Inertia.—If the moments of inertia with respect to any set of rectangular axes are known, it is sometimes convenient to be able to rotate the axes through some angle θ to a new position. The determination of the moments of inertia with respect to the new set of axes involves terms of the form $\int x\, y\, dA$. These terms are called products of inertia.

The term $H = \int x\, y\, dA$ represents, as indicated in Fig. 362, the product of an elementary area and its distance from each of the inertia axes.

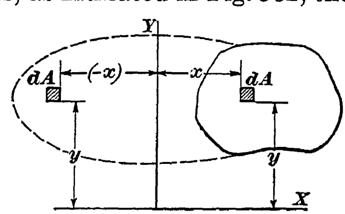


FIG. 362

It is easily seen that, if the area extends into more than one quadrant, as indicated by the dotted lines in Fig. 362, the result-

ant product may have either the plus sign or the negative sign, the proper one depending on the arrangement of the area. *If all of the area is in the first or third quadrant, the sign will be plus; if it is in the second or fourth quadrant, the sign will be negative because either x or y will be negative.*

Product of inertia has the same units as moment of inertia, namely, inches or feet to the fourth power.

EXAMPLE

Determine the product of inertia of the triangle shown in Fig. 363 with respect to the axes indicated.

From similiar triangles,

$$\frac{y}{x}=\frac{6}{9} \text{ and } y=\frac{2}{3}x$$

$$H=\int x\,y\,dA=\int_0^9\int_0^{\frac{2}{3}x} x\,y\,dx\,dy$$

$$H=\frac{2}{9}\int_0^9 x^3\,dx=364.5 \text{ in.}^4$$

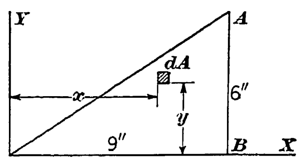


FIG. 363

PROBLEMS

395. Determine the product of inertia for a $6''\times 8''$ rectangle in the first quadrant, if the sides of the rectangle are the X and Y axes. *Ans.* *576 in.*^{4}

396. Compute the product of inertia for a quadrant of a circle if the bounding radii are used as axes and the area is assumed to be in the second quadrant. *Ans.* $-\frac{r^4}{8}$.

397. Calculate H for the area under the parabola in Fig. 364.

398. Determine by integration the product of inertia of the triangle in Fig. 365 with respect to the X_0 and Y_0 axes.

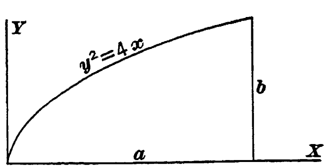


FIG. 364

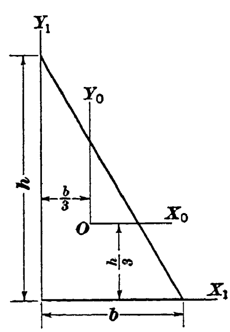


FIG. 365

101. Effect of Axes of Symmetry on Product of Inertia.—If the axes are so selected that either or both are axes of symmetry for the area involved, the product of inertia will become zero.

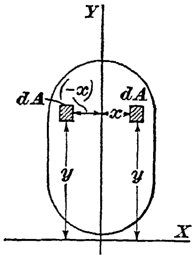


FIG. 366

In Fig. 366 is represented an oval area which is symmetrical with respect to the Y axis. It is easily seen that, for an area so arranged, for each elementary area dA producing a positive $\int x\,y\,dA$ term there is an equal area dA which will produce a negative $\int x\,y\,dA$ term. When these are added, the result will be zero for the product of inertia of the entire area.

102. Parallel Axis Theorem for Product of Inertia.—If the product of inertia, H, is known for any set of rectangular axes through the centroid, the product of inertia for any set of parallel axes may be found in a manner similar to that used in the transfer of moments of inertia.

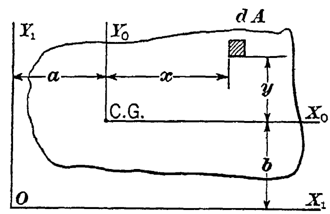


FIG. 367

In Fig. 367 let X_0 and Y_0 be the centroidal axes of the area shown. Also, let X_1 and Y_1 be any other parallel axes.

$$H_{x_1y_1} = \int (x+a)\,(y+b)\,dA$$

$$H_{x_1y_1} = \int x\,y\,dA + a\int y\,dA + b\int x\,dA + a\,b\int dA$$

Since $\int y\,dA = 0$, $\int x\,dA = 0$, and $\int x\,y\,dA = H_0$,

$$H_{x_1y_1} = H_0 + a\,b\,A$$

The quantities a and b may be positive or negative, the sign depending on the location of the X_1 and Y_1 axes with reference to the centroid of the area A. If the centroid is in the first or third quadrant, the term abA will be positive. If the centroid is in the second quadrant, the quantity a will be negative and b will be positive. If the centroid is in the fourth quadrant, a will be positive and b will be negative.

If either or both of the centroidal axes is an axis of symmetry, the product of inertia H_0 is zero. Then, since $H_{x_1y_1} = H_0 + a\,b\,A$, it follows that the product of inertia $H_{x_1y_1}$ with respect to any pair of axes parallel to the centroidal axes will always be $H_{x_1y_1} = a\,b\,A$, where a and b may be positive or negative.

Sometimes it is possible to divide an area into component areas each of which has an axis of symmetry through its centroid and parallel to one of the axes with respect to which the product of inertia of the total area is desired. For such an area the desired product of inertia can be obtained by applying the parallel axis theorem to each of the component areas in turn.

EXAMPLE

Determine $H_{x_1y_1}$ for the area in Fig. 368.

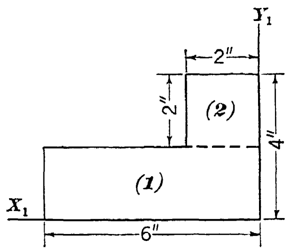


FIG. 368

Divide the area into the component areas (1) and (2), each of which has an axis of symmetry that is parallel to one of the reference axes.

Because of the symmetry, the products of inertia of areas (1) and (2) with respect to axes through their centroids parallel to the X_1 and Y_1 axes are zero. Hence, $H_0 = 0$, and

$$H_{x_1y_1} = a\,b\,A$$

For area (1)... $H_{x_1y_1} = (-3)\,(+1)\,(6)\,(2) = -36$

For area (2)... $H_{x_1y_1} = (-1)\,(+3)\,(2)\,(2) = -12$

For the composite area,

$$H_{x_1y_1} = -48 \text{ in.}^4$$

PROBLEMS

399. Solve Problem 395 by the parallel axis theorem and without integration.

400. Calculate H_0 for the quadrant of Problem 396, Art. 100. *Ans.* *0.0165* r^4 *in.*^{4}

401. For the triangular area in Fig. 363, calculate the product of inertia with respect to axes through vertex A parallel to the given axes. Calculate the product of inertia also for axes through point B.

402. Determine the product of inertia H_0 for the triangle in Fig. 365 by computing $H_{x_1y_1}$ by integration and then using the parallel axis theorem.

403. Determine the product of inertia of a $8'' \times 6'' \times 1''$ angle section with respect to axes that are tangent to the 6-in. and 8-in. edges. Also, compute H_0 for the parallel axes through the centroid of the section. *Ans.* $H_{x_1y_1} = \pm 24.75$ *in.*^{4}; $H_0 = \pm 32.09$ *in.*^{4}

103. Relation Between Moments of Inertia With Respect to Two Sets of Rectangular Axes Through the Same Point.

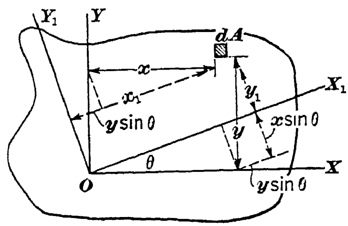


FIG. 369

—If OX and OY, Fig. 369, are any set of rectangular axes, and OX_1 and OY_1 are any other set of axes making an angle θ with OX and OY, the moments of inertia with respect to OX_1 and OY_1 may be found in the following manner.

$$I_{x_1} = \int (y_1)^2\, dA = \int (y \cos\theta - x \sin\theta)^2\, dA$$
$$I_{x_1} = \cos^2\theta \int y^2\, dA + \sin^2\theta \int x^2\, dA - 2 \sin\theta \cos\theta \int x\, y\, dA$$
$$I_{x_1} = I_x \cos^2\theta + I_y \sin^2\theta - H_{xy} \sin 2\theta \qquad (1)$$
$$I_{y_1} = \int (x_1)^2\, dA = \int (x \cos\theta + y \sin\theta)^2\, dA$$
$$I_{y_1} = I_y \cos^2\theta + I_x \sin^2\theta + H_{xy} \sin 2\theta \qquad (2)$$
$$\cos^2\theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \text{ and } \sin^2\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

Equations (1) and (2) may also be written in the following manner:

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H_{xy} \sin 2\theta \qquad (1')$$

$$I_{y_1} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + H_{xy} \sin 2\theta \qquad (2')$$

Adding equations (1) and (2) gives

$$I_{x_1} + I_{y_1} = I_x (\sin^2\theta + \cos^2\theta) + I_y (\sin^2\theta + \cos^2\theta)$$

or

$$I_{x_1} + I_{y_1} = I_x + I_y$$

Since, by Art. 96, $I_{x_1} + I_{y_1} = J$, it follows that $I_x + I_y = J$. This relationship leads to the following important statement. *The sum*

of the moments of inertia with respect to any pair of rectangular axes is equal to the sum of the moments of inertia with respect to any other pair of rectangular axes through the same point.

104. Relation Between Products of Inertia for Two Sets of Rectangular Axes Through the Same Point.—The relation between the products of inertia with respect to two sets of axes may be established in a manner similar to that used in Art. 103 for moments of inertia. For the conditions represented in Fig. 369,

$$H_{x_1y_1}=\int x_1\,y_1\,dA=\int(x\cos\theta+y\sin\theta)\,(y\cos\theta-x\sin\theta)\,dA$$
$$=(\cos^2\theta-\sin^2\theta)\int x\,y\,dA+\sin\theta\cos\theta\int(y^2-x^2)\,dA$$
$$H_{x_1y_1}=H_{xy}\cos 2\theta+\frac{1}{2}\,(I_x-I_y)\sin 2\theta$$

Thus, if the product of inertia and the moments of inertia for any set of rectangular axes through a point are known, the product of inertia with respect to any other set of rectangular axes through the same point may be found.

EXAMPLE 1

Determine the moment of inertia of a square, Fig. 370, with respect to a diagonal.

$$I_x=\frac{a^4}{3};\ I_y=\frac{a^4}{3};\ H_{xy}=\int_0^a\int_0^a x\,y\,dx\,dy=\frac{a^4}{4}$$

$$I_{x_1}=I_x\cos^2\theta+I_y\sin^2\theta-H_{xy}\sin 2\theta$$
$$I_{x_1}=\frac{a^4}{6}+\frac{a^4}{6}-\frac{a^4}{4}=\frac{a^4}{12}$$

EXAMPLE 2

Solve for the product of inertia of the square shown in Fig. 370 with respect to the X_1 and Y_1 axes.

$$H_{x_1y_1}=H_{xy}\cos 2\theta+\frac{1}{2}\,(I_x-I_y)\sin 2\theta$$

$$=\frac{a^4}{4}\times 0+\frac{1}{2}\left(\frac{a^4}{3}-\frac{a^4}{3}\right)1=\ 0$$

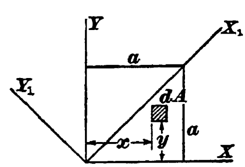


FIG. 370

Explain the result just obtained.

PROBLEMS

404. Compute the moment of inertia of the square in Fig. 370 with respect to the Y_1 axis. *Ans.* $\frac{7}{12}a^4$ *in.*^{4}

405. Determine the moment of inertia of a quadrant of a circle with respect to the bisecting radius, or axis X_1 in Fig. 371. The radius is 3 in. Prove the validity of the answer by applying the relationship developed in Art. 103.

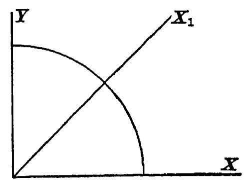
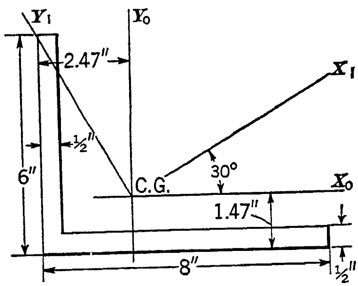


FIG. 371



$I_{y_0}=44.31$ in.^{4} $I_{x_0}=21.68$ in.^{4}

FIG. 372

406. Solve for the product of inertia with respect to the X_0 and Y_0 axes, and also the X_1 and Y_1 axes, for the angle section shown in Fig. 372.

407. Compute the moments of inertia of the angle in Fig. 372 with respect to the X_1 and Y_1 axes. *Ans. 43.21 in.*^{4}*; 22.86 in.*^{4}

105. Maximum and Minimum Moments of Inertia.—The principal axes of an area are the two rectangular axes through any given point in the area, with respect to which *the moments of inertia have, respectively, a maximum value and a minimum value* when compared with the moments of inertia about any other pair of rectangular axes through the same point. For every point in a given area there is a pair of rectangular axes for which the moments of inertia are either larger or smaller than those for any other set of axes through the same point. Generally the principal axes for which the moments of inertia are desired are those which pass through the centroid of the area.

The moment of inertia of any area with respect to an axis making an angle θ with some other axis is given by equation (1) or equation (1′), Art. 103. If this equation is differentiated with respect to θ and the first derivative is set equal to zero, the value

of θ for which I_{x_1} has a maximum or minimum value may be determined.

$$I_{x_1}=I_x \cos^2\theta+I_y \sin^2\theta-H_{xy}\sin 2\theta$$

$$I_{x_1}=I_x\frac{1+\cos 2\theta}{2}+I_y\frac{1-\cos 2\theta}{2}-H_{xy}\sin 2\theta$$

$$=\frac{I_x+I_y}{2}+\frac{I_x-I_y}{2}\cos 2\theta-H_{xy}\sin 2\theta$$

$$\frac{dI_{x_1}}{d\theta}=(I_y-I_x)\sin 2\theta-2H_{xy}\cos 2\theta$$

When the right side of this equation is equated to zero, solution of the resulting equation gives

$$\tan 2\theta=\frac{2H_{xy}}{I_y-I_x}$$

This relation shows that there are two values of 2θ, which differ by 180°, and thus there are two values of θ which are 90° apart. One value of θ locates the axis of maximum moment of inertia, and the other locates that of minimum moment of inertia. *These are the principal axes.*

If either the X axis or the Y axis is an axis of symmetry, then $H_{xy}=0$, as was shown in Art. 101. In such a case, $\tan 2\theta=0$ and $\theta=0$ or 90°. This result indicates that the axis of symmetry is one of the principal axes and the other principal axis is perpendicular to the axis of symmetry. *Axes of symmetry are always principal axes.*

EXAMPLE

Compute the moments of inertia of the $8''\times 6''\times\frac{1}{2}''$ angle in Fig. 372 with respect to the principal axes through the centroid.

$$\tan 2\theta=\frac{2(-18.33)}{44.31-21.68}=-1.619$$

$$2\theta=121.7°$$

$$\theta=60.9° \text{ and } \theta+90°=150.9°$$

$H_{x_0y_0}=-18.33$; $I_{x_0}=21.68$; $I_{y_0}=44.31$. For $\theta=60.9°$

$$I_{x_1}=I_x\cos^2\theta+I_y\sin^2\theta-H_{xy}\sin 2\theta$$

$$=21.68\ (0.4871)^2+44.31\ (0.8734)^2-(-18.33)\ 0.8508$$

$$I_{x_1}=54.55 \text{ in.}^4$$

$$I_{y_1}=I_y\cos^2\theta+I_x\sin^2\theta+H_{xy}\sin 2\theta$$

$$=44.31\ (0.4871)^2+21.68\ (0.8734)^2+(-18.33)\ 0.8508$$

$$I_{y_1}=11.4 \text{ in.}^4$$

PROBLEMS

408. Determine the maximum and minimum moments of inertia for a 4″×8″ rectangle with respect to axes which intersect at the centroid of the rectangle. *Ans. 170.6 in.*^{4}*; 42.6 in.*^{4}

409. Determine the maximum and minimum moments of inertia for an 8″×8″×$\frac{3}{4}$″ angle section with respect to axes through the centroid.

410. Compute the maximum and the minimum moments of inertia for a 6″×4″×$\frac{1}{2}$″ angle section with respect to axes through the centroid of the section.

411. Determine the maximum and minimum centroidal moments of inertia for a 8″×5″×1″ Z-bar section.

REVIEW PROBLEMS

412. Calculate I_{x_0}, I_{y_0}, k_{x_0}, and k_{y_0} for the area shown in Fig. 373. *Ans. 136 in.*^{4}*; 103.8 in.*^{4}*; 1.89 in.; 1.65 in.*

413. Calculate I_{y_0} for an ellipse which has its major axis coincident with the X_0 axis. The equation of the ellipse is $b^2x^2+a^2y^2=a^2b^2$.

414. Compute I_{x_0} for the section shown in Fig. 374.

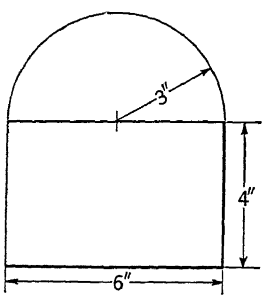


FIG. 373

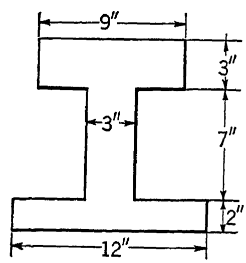


FIG. 374

415. Determine the centroidal moments of inertia for the angle section shown in Fig. 375 with respect to axes parallel to the legs of the angle.

416. Locate the centroid, and calculate I_{x_0} and I_{y_0} for the T section shown in Fig. 376. *Ans. 136 in.*^{4}*; 40 in.*^{4}

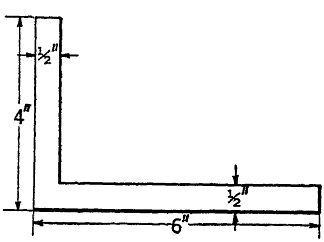


FIG. 375

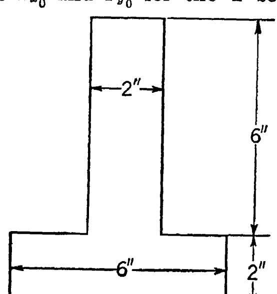


FIG. 376

417. Locate the centroid, and calculate I_{x_0} for the area shown in Fig. 377.

418. Locate the centroid of the area shown in Fig. 378, and compute the moments of inertia with respect to the X_1 and X_0 axes.

419. Determine the least radius of gyration for the beam section shown in Fig. 379. For one channel, $I_{11} = 115.2$ in.^{4}; $I_{22} = 4.6$ in.^{4}; $A = 10.27$ sq in.; $y = 0.69$ in.

420. Calculate I_{x_0}, I_{y_0}, and the least radius of gyration for the section in Fig. 380. *Ans.* $I_x = 12{,}396$ *in.*^{4}; $I_y = 4{,}581.2$ *in.*^{4}; *4.95 in.*

421. Calculate I_{x_0}, I_{y_0}, and the least radius of gyration for the section in Fig. 381. For one angle, $I = 31.9$ *in.*^{4}; $A = 9.73$ *in.*^{2}; $x = 1.82$ *in.*

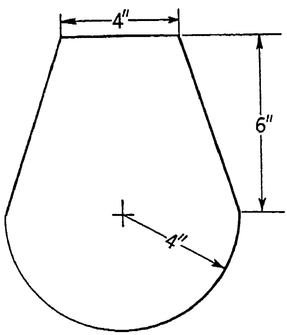


FIG. 377

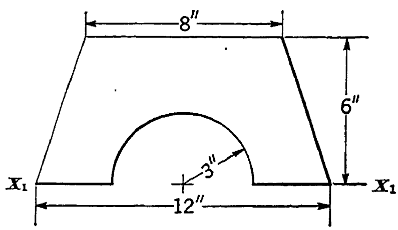


FIG. 378

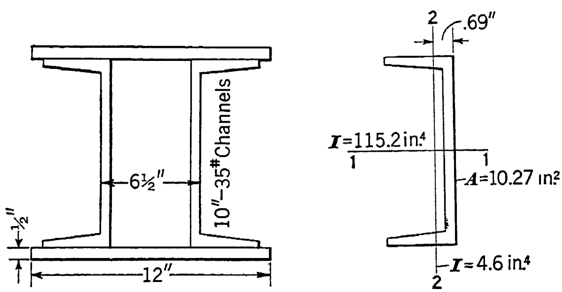


FIG. 379

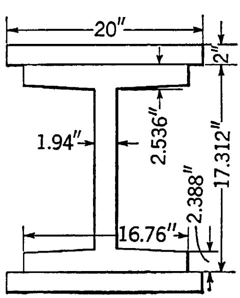


FIG. 380

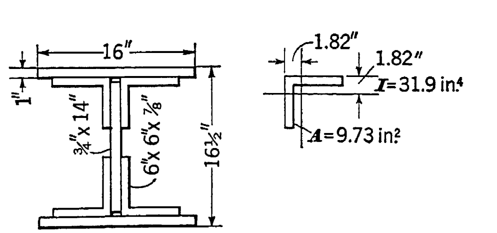


FIG. 381

422. Determine I_{11} and I_{22} for the Z bar shown in Fig. 382.

423. Compute the polar moment of inertia of the section shown in Fig. 383 with respect to an axis through the centroid.

424. Determine the moment of inertia of the sector shown in Fig. 384 with respect to the tangent AB. *Ans.* $955\ in.^4$

425. What is the polar moment of inertia for the area shown in Fig. 384 with respect to an axis through the point C?

426. Determine the maximum and minimum moments of inertia for axes passing through the center of gravity of an $8'' \times 8'' \times \frac{3}{4}''$ angle section.

427. Compute the maximum and minimum centroidal moments of inertia for the Z bar shown in Fig. 382. *Ans.* $17.63\ in.^4$; $1.91\ in.^4$

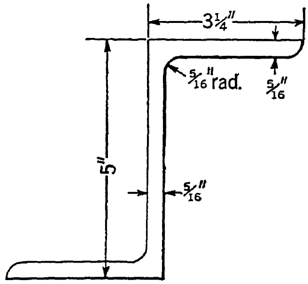


Fig. 382

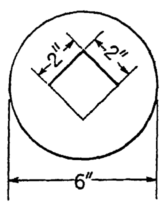


Fig. 383

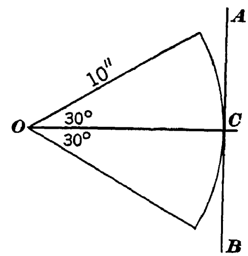


Fig. 384

428. Determine the maximum and minimum centroidal moments of inertia for the angle section of Fig. 375.

429. Compute the maximum and minimum centroidal moments of inertia for the section shown in Fig. 374.

430. Fig. 385 is a cross-section of a concrete column. Determine the maximum and minimum centroidal moments of inertia and the least radius of gyration of the section.

431. Calculate the product of inertia of the channel section shown in Fig. 386 with respect to its X_0 and Y_0 axes.

432. Determine the moment of inertia of the channel in Fig. 386 with respect to an axis through the centroid at an angle of 30° with the horizontal.

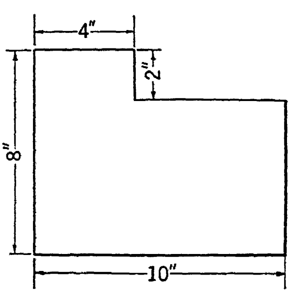


Fig. 385

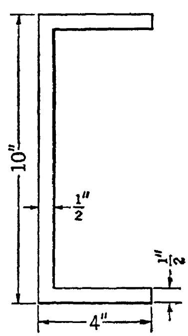


Fig. 386

CHAPTER 12

SECOND MOMENTS OF MASS—MOMENTS OF INERTIA OF SOLIDS

106. **General Discussion.**—In the study of the rotation o solid bodies, certain fundamental relationships are developed, suc as Resultant Torque$=I\alpha$ and Kinetic Energy of Rotatio (K.E.)$=\frac{1}{2}I\omega^2$. The quantity I in these equations represents term whose original form was $\int \rho^2\,dM$, where ρ represents th distance to the differential mass dM from the axis about which th entire body is turning. The expression $\int \rho^2\,dM$ is similar to th expression $\int x^2\,dA$ for the moment of inertia of area. It was firs encountered about 1673 by Huygens, the Dutch Archimedes, i his study of the compound pendulum.

The moment of inertia of a solid has a physical significanc which the moment of inertia of a plane area does not posses This significance will be more clearly visualized during the stud of rotation. For the present it is sufficient to state that observa tion of rotating objects teaches us that they tend to continue t rotate at a constant speed unless acted upon by external force Practical experience teaches us that the resistance offered to an change in the motion is directly proportional to the magnitude o the mass involved and the square of its distance from the axis o rotation.

When $I=\int \rho^2\,dM$ is to be evaluated for a given body,

$$M=\gamma\,V;\ dM=\gamma\,dV;\ I=\gamma\int \rho^2\,dV$$

Here γ is the density of the body, or the number of units of mas per unit volume, and ρ is the distance, in feet, to each element o volume from the axis of reference. Distances are expressed i feet because in engineering calculations mass is $M=\dfrac{W}{g}$, where W i in pounds and g is the acceleration of gravity in feet per secon squared.

No name has been given to the unit of moment of inerti because it is a derived unit, which is a combination of the units o force, length, and time, as shown in the following equation:

$$M=\frac{W}{g}=\frac{\text{pounds}\times\text{seconds}^2}{\text{feet}}=\text{slugs}$$

$$I=\int\rho^2\,dM=\frac{\text{feet}^2\times\text{pounds}\times\text{seconds}^2}{\text{feet}}=\text{slugs-feet}^2$$

The radius of gyration of a body, with respect to any axis, is the distance from the axis at which the entire mass of the body could be concentrated and still have the same moment of inertia. Thus,

$$I=Mk^2 \text{ and } k=\sqrt{\frac{I}{M}}$$

where k is the radius of gyration.

107. Moments of Inertia of Solids by Integration.—Fortunately, many of the solids for which it is necessary to determine the moment of inertia are of rather simple form or can be divided into parts each of which falls into this classification.

The moment of inertia of each elementary part is obtained by setting up integral expressions of the form $\int\rho^2\,dM$. It is necessary that the element of mass be selected according to one of the following rules.

1. All parts of the element must be the same distance from the axis with respect to which the moment of inertia of the body is desired (see Example 1, Art. 108).
2. Select the element so that its moment of inertia is known for the axis with respect to which the moment of inertia of the entire body is desired. The resultant moment of inertia is then found by integration between the proper limits (see Example 2, Art. 108).
3. The element may be selected so that its moment of inertia with respect to an axis through its own center of gravity and parallel to the desired axis is known. The moment of inertia of the entire body may then be obtained by applying the transfer theorem to each element and integrating (see Example 4, Art. 110).

108. Moments of Inertia of Elementary Solids.—In the examples which follow, γ is the mass per unit volume and w is the weight of a cubic foot of the material. Then,

$$\gamma=\frac{w}{g}$$

For cast-iron, $\gamma=\frac{450}{32.2}$; for steel, $\gamma=\frac{490}{32.2}$.

EXAMPLE 1

Determine the moment of inertia of the right circular cylinder shown in Fig. 387 with respect to its geometric axis.

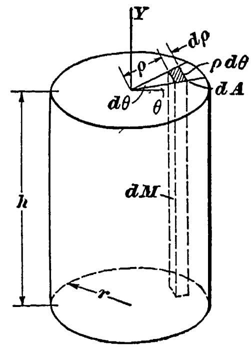


FIG. 387

First Solution:

$$I_y = \int \rho^2 \, dM$$
$$dM = \gamma \, h \, dA$$
$$I_y = \gamma \, h \int \rho^2 \, dA$$

$\int \rho^2 \; dA = J =$ the polar moment of inertia of the cross-section

$$I_y{}^* = \gamma \, h \, J$$

$$I_y = \frac{\gamma \, h \, \pi \, r^4}{2}; \; J = \frac{\pi \, r^4}{2}; \; M = \gamma \, \pi \, r^2 \, h$$

$$I_y = \frac{M \, r^2}{2}.$$

Second Solution:

$$dM = \gamma \, h \, d\rho \, \rho \, d\theta$$

$$I_y = \gamma \, h \int_0^{2\pi} \int_0^r \rho^3 \, d\rho \, d\theta$$

$$I_y = \frac{\gamma \, h \, 2 \, \pi \, r^4}{4} = \frac{\gamma \, h \, \pi \, r^4}{2}$$

$$I_y = \frac{M \, r^2}{2}$$

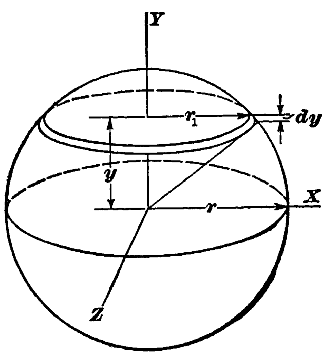


FIG. 388

EXAMPLE 2

Obtain the moment of inertia of a sphere with respect to a diameter.

Apply Rule 2, Art. 107. Take the slice indicated in Fig. 388 as the element of mass. This slice may be considered to be a small right cylinder. By Example 1, the moment of inertia of the slice with respect to its geometric axis is $\frac{1}{2} \, dM \, r_1^2$. Then I_y for the sphere is:

* The expression $I_y = \gamma h J$ is a general form, which may be applied to any right prism if the polar moment of inertia J of the cross-section is known.

$$I_y = 2\int_0^r \frac{1}{2}\,dM\,r_1^2 \text{ and } dM = \gamma\,\pi\,r_1^2\,dy$$

$$I_y = \gamma\,\pi \int_0^r r_1^4\,dy$$

$$r_1^2 = r^2 - y^2$$

$$I_y = \gamma\,\pi \int_0^r (r^2 - y^2)^2\,dy$$

$$I_y = \frac{8}{15}\,\gamma\,\pi\,r^5 \text{ and } M = \gamma\,\frac{4}{3}\,\pi\,r^3$$

$$I_y = \frac{2}{5}\,M\,r^2$$

EXAMPLE 3

Determine the moment of inertia of a right circular cone with respect to a geometric axis.

Apply Rule 2, Art. 107. As shown in Fig. 389, the element of mass is again the thin slice of radius r_1 and height dy. The moment of inertia of the slice with respect to the geometric axis is $\frac{1}{2}\,dM\,r_1^2$ and the moment of inertia of the cone is

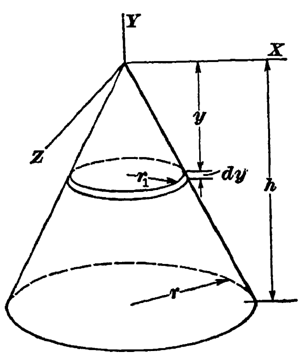


Fig. 389

$$I_y = \int \frac{1}{2}\,dM\,r_1^2$$

$$dM = \gamma\,\pi\,r_1^2\,dy$$

$$I_y = \frac{1}{2}\,\gamma\,\pi \int r_1^4\,dy$$

$$\frac{r_1}{r} = \frac{y}{h} \text{ and } r_1 = \frac{r\,y}{h}$$

$$I_y = \frac{1}{2}\,\gamma\,\pi\,\frac{r^4}{h^4}\int_0^h y^4\,dy$$

$$I_y = \frac{1}{2}\,\gamma\,\frac{\pi\,r^4}{h^4}\,\frac{h^5}{5} = \frac{1}{10}\,\gamma\,\pi\,r^4\,h \text{ and } M = \gamma\,\frac{\pi\,r^2\,h}{3}$$

$$I_y = \frac{3}{10}\,M\,r^2$$

EXAMPLE 4

Determine the moment of inertia of a slender rod, Fig. 390, with respect to an axis through its center of gravity and perpendicular to the length.

Let the rod have a cross-sectional area A and a length L.

$$I_y=\int_{-\frac{L}{2}}^{\frac{L}{2}}\gamma\, l^2\, A\, dl=\left[\frac{\gamma\, A\, l^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}=\frac{\gamma\, A\, L^3}{12}$$

$$I_y=\frac{M\,L^2}{12}$$

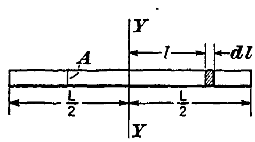


FIG. 390

PROBLEMS

433. Solve for the moment of inertia of a right square prism with sides a and height h with respect to a geometric axis perpendicular to the base. *Ans.* $\frac{\gamma h a^4}{6}$.

434. Derive the moment of inertia of a slender rod with respect to an axis through one end perpendicular to the rod.

435. In Problem 434 let the axis make an angle θ with the length of the rod. Determine I for this axis.

436. What is the moment of inertia of a hemisphere with respect to the radius which is perpendicular to the plane of the base? *Ans.* $\frac{2}{5}\,M\,r^2$.

437. Explain the answer to Problem 436.

438. Solve Example 3 by using a hollow cylinder as the element of mass.

439. Derive the expression for the moment of inertia of a hollow, right, circular cylinder with an outside radius r_1 and an inside radius r_2 with respect to the geometric axis of the cylinder.

109. The Transfer Formula for Moment of Inertia of Mass. If the moment of inertia with respect to an axis through the centroid of a mass is known, the moment of inertia with respect to any other parallel axis may be determined by applying the relationship which will now be derived.

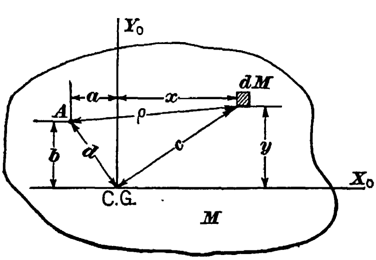


FIG. 391

Fig. 391 represents a plane perpendicular to any axis through the center of gravity of any mass M. The X_0 and Y_0 axes are perpendicular to this axis and

also pass through the center of gravity of the mass M. Some other axis which is parallel to the given axis passes through the plane at point A. The moment of inertia of the mass M with respect to the axis through A is

$$I_A = \int \rho^2 \, dM$$
$$\rho^2 = (x+a)^2 + (y-b)^2$$
$$I_A = \int (x^2 + 2ax + a^2 + y^2 - 2by + b^2) \, dM$$
$$x^2 + y^2 = c^2 \text{ and } a^2 + b^2 = d^2$$
$$I_A = \int c^2 \, dM + \int d^2 \, dM + 2a \int x \, dM - 2b \int y \, dM$$
$$I_A = I_0 + M \, d^2$$

The student should note that this theorem applies only when one of the two parallel axes passes through the center of gravity of the mass M. The theorem just stated may be transformed into the following form:

$$M \, k_A^2 = M \, k_0^2 + M \, d^2$$
$$k_A^2 = k_0^2 + d^2$$

EXAMPLE

Solve for the moment of inertia of a sphere with respect to a tangent.

From Example 2, Art. 108,

$$I_0 = \frac{2}{5} M \, r^2$$
$$I_t = \frac{2}{5} M \, r^2 + M \, r^2 = \frac{7}{5} M \, r^2$$

PROBLEMS

440. Check the result of Problem 434 by means of the transfer formula.

441. Determine the moment of inertia of the cone shown in Fig. 389 with respect to an axis parallel to the geometric axis and tangent to the base.

442. Determine the moment of inertia of a right circular cylinder with respect to an element of the curved surface.

443. Solve for the moment of inertia of a hemisphere with respect to a tangent parallel to the diametral plane.

110. Moments of Inertia of Certain Thin Plates.—It is often necessary to determine the moment of inertia of parts of structures which have been fabricated from thin plates. The following examples will illustrate the technique for obtaining the moments of inertia of certain standard shapes.

EXAMPLE 1

Derive the expressions for I_{x_0}, I_{y_0}, and J_0 for the thin circular plate of thickness t shown in Fig. 392. The polar axis is normal to the plate through the centroid at O.

$$J_0 = \int \rho^2\, dM = \int \rho^2\, \gamma\, t\, dA$$

But
$$\int \rho^2\, dA = \frac{\pi r^4}{2}$$

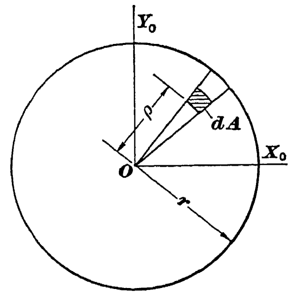


FIG. 392

$$J_0 = \gamma\, t\, \frac{\pi r^4}{2} \text{ and } M = \gamma\, t\, \pi r^2$$

$$J_0 = \frac{M r^2}{2}$$

$$I_{x_0} + I_{y_0} = J_0$$

$$2 I_{x_0} = \frac{M r^2}{2}$$

$$I_{x_0} = I_{y_0} = \frac{M r^2}{4}$$

EXAMPLE 2

Develop the expressions for the moments of inertia of a triangular plate, Fig. 393, of base b, altitude h, and thickness t with respect to axes through the vertex and the centroid parallel to the base and also with respect to the base.

$$I_{x_1} = \int y^2\, dM = \int y^2\, \gamma\, t\, u\, dy$$

$$\frac{u}{b} = \frac{y}{h} \text{ and } u = \frac{b}{h}\, y$$

$$I_{x_1} = \gamma\, t\, \frac{b}{h} \int_0^h y^3\, dy = \frac{\gamma\, t\, b\, h^3}{4} \text{ and } M = \frac{\gamma\, t\, b\, h}{2}$$

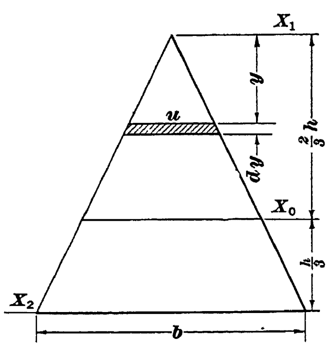


FIG. 393

$$I_{x_1} = \frac{M h^2}{2}$$

$$I_{x_1} = I_{x_0} + M d^2$$

$$I_{x_0} = \frac{M h^2}{2} - M\left(\frac{2}{3} h\right)^2 = \frac{M h^2}{18}$$

$$I_{x_2} = I_{x_0} + M d^2$$

$$I_{x_2} = \frac{M h^2}{18} + M\left(\frac{h}{3}\right)^2 = \frac{M h^2}{6}$$

EXAMPLE 3

Develop formulas for the moments of inertia I_x, I_y, and J of a thin elliptical plate of thickness t with respect to the centroidal axes shown in Fig. 394 (*a*).

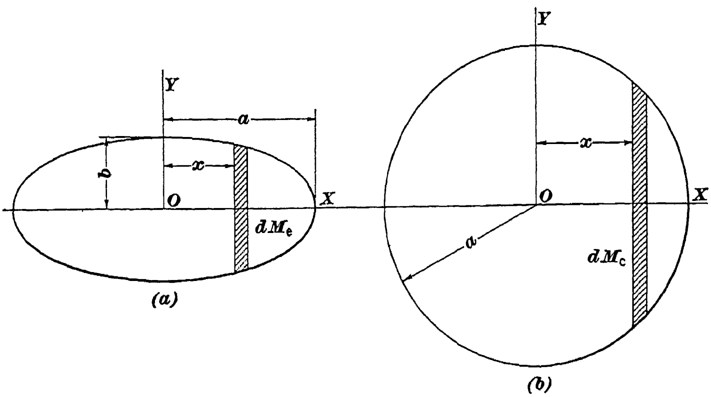


FIG. 394

Fig. 394 (*b*) represents a circular plate also of thickness t and with a radius a equal to the semi-major axis a in Fig. 394 (*a*). The masses of the plates are given by the following formulas:

$$M_e = \gamma\, t\, \pi\, a\, b \text{ and } M_c = \gamma\, t\, \pi\, a^2$$

$$M_e = M_c \frac{b}{a}$$

For the ellipse, $I_y = \int x^2\, dM_e$; and, for the circle, $I_y = \int x^2\, dM_c$. The moments of inertia are therefore proportional to the masses.

$$I_{y_e} = I_{y_c} \frac{b}{a}$$

But $I_{y_c} = \frac{M_c\, a^2}{4}$ from Example 1. Therefore,

$$I_{y_e} = \frac{\gamma\, t\, \pi\, a^2}{4} a^2 \frac{b}{a} = \frac{\gamma\, t\, \pi\, a^3\, b}{4} \text{ and } M_e = \gamma\, t\, \pi\, a\, b$$

$$I_{y_e} = M_e \frac{a^2}{4}$$

$$I_{x_e} = M_e \frac{b^2}{4}$$

$$J_0 = M_e \frac{a^2}{4} + M_e \frac{b^2}{4} = \frac{1}{4} M_e\, (a^2 + b^2)$$

EXAMPLE 4

By application of Rule 3, Art. 107, compute the moment of inertia of a right circular cone with respect to an axis through the vertex and parallel to the base.

The element of area selected is the thin slice indicated in Fig. 395. The moment of inertia of this disk of radius r_1 and thickness dy with respect to an axis through the center of gravity of the disk and parallel to the X axis is $\frac{1}{4}\,dM\,r_1^2$, as shown in Example 1.

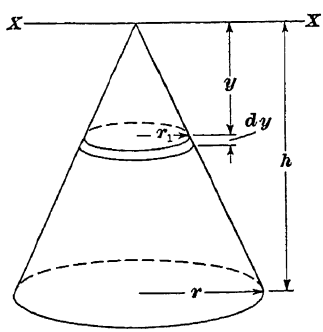


FIG. 395

The moment of inertia of the disk with respect to the X axis may be found by applying the transfer formula as follows:

$$I_x=\int\frac{1}{4}\,dM\,r_1^2+\int y^2\,dM$$

The moment of inertia of the entire cone with respect to the X axis will then be

$$I_x=\int\frac{1}{4}\,dM\,r_1^2+\int y^2\,dM$$

$$\frac{r_1}{r}=\frac{y}{h};\; r_1=\frac{r\,y}{h};\; dM=\gamma\,\pi\,r_1^2\,dy$$

$$I_x=\frac{1}{4}\,\gamma\,\frac{\pi\,r^4}{h^4}\int_0^h y^4\,dy+\gamma\,\frac{\pi\,r^2}{h^2}\int_0^h y^4\,dy$$

$$I_x=\frac{1}{4}\,\gamma\,\frac{\pi\,r^4}{h^4}\,\frac{h^5}{5}+\gamma\,\frac{\pi\,r^2}{h^2}\,\frac{h^5}{5}$$

$$I_x=\frac{3}{5}\,M\left(\frac{r^2}{4}+h^2\right)$$

PROBLEMS

444. Derive expressions for the moments of inertia for a rectangular plate of base b, height h, and thickness t with respect to its base and a centroidal axis parallel to the base.

445. A semicircular plate has a radius r and a thickness t. Derive the expression for the moment of inertia with respect to the diameter.

446. Determine the polar moment of inertia for the plate of Problem 445 with respect to an axis normal to the diameter at its mid-point.

447. Find the moment of inertia of a right circular cone with respect to an axis through the center of gravity and parallel to the base. *Ans.* $\frac{3}{5}M\frac{4r^2+h^2}{16}$.

448. Determine the moment of inertia of a right circular cylinder of radius r and height h with respect to a diameter of the base.

449. Show that the moment of inertia of the cylinder of Problem 448 with respect to a centroidal axis parallel to the base is $M\frac{3r^2+h^2}{12}$.

450. By using the result of Problem 448, determine the moment of inertia of a slender rod of length L with respect to an axis through one end perpendicular to the rod.

451. Explain why the answers to Problems 434 and 450 are different.

111. Moments of Inertia of Composite Bodies; Units in Moment of Inertia.—As stated in Art. 107, many of the bodies for which the engineer requires the moment of inertia are simple geometric figures, for which the moments of inertia can be obtained by calculus. Some, however, are combinations of several geometric units. The moment of inertia of such a body is obtained by determining the moment of inertia of each of the several parts with respect to any desired axis, and then adding these to get the resultant moment of inertia. Where certain parts of the body are cut away, the moments of inertia of these parts are deducted from the resultant sum.

The matter of units is very important. Since in engineering the foot-pound-second system of units is used, *it is necessary that all dimensions be expressed in feet, weights in pounds, and g in feet per second*^{2}.

EXAMPLE

If the sphere of Example 1, Art. 109, is made of cast iron and has a radius of 18 in., what are its moment of inertia and its radius of gyration?

$$I=\frac{7}{5}M\,r^2;\ M=\frac{450}{32.2}\times\frac{4}{3}\,\pi\,1.5^3=197 \text{ slugs}$$

$$I=\frac{7}{5}\times 197\times 1.5^2=622 \text{ pounds}\times\text{feet}\times\text{seconds}^2 \text{ or slugs}\times\text{feet}^2$$

$$k=\sqrt{\frac{I}{M}}=\sqrt{\frac{622}{197}}=1.77 \text{ ft}$$

PROBLEMS

452. Compute the moment of inertia of a steel cylinder with respect to its geometric axis. The cylinder is 18 in. in diameter and 1 ft long.

453. Compute the moment of inertia of a cast-iron sphere 9 in. in diameter with respect to a tangent to its surface.

454. Determine the moment of inertia of the cylinder of Problem 452 with respect to a centroidal axis parallel to the base.

455. If the sphere in Problem 453 is hollow and has an inside diameter of 6 in., what is its moment of inertia with respect to an axis that has an eccentricity of 4 in.?

456. A cast-iron pulley has a diameter of 18 in., a face 6 in. wide, a rim 2 in. thick, and a web 1 in. thick. Neglecting the pulley hub, determine the moment of inertia with respect to the geometric axis.

REVIEW PROBLEMS

457. Using the formula developed in Example 1, Art. 108, determine the moment of inertia of a right square prism with sides a and height h with respect to an edge of the prism that is parallel to the dimension h. *Ans.* $\frac{2}{3}Ma^2$.

458. What are the moment of inertia and the radius of gyration of a cast-iron disk, which is 24 in. in diameter and 6 in. thick, with respect to its geometric axis?

459. Determine the moment of inertia of the disk in Problem 458 for an axis parallel to the geometric axis and 10 in. away.

460. A cast-iron ball of 6-in. radius is fastened to the end of a steel rod 1 in. in diameter and 4 ft long. Find the moment of inertia for an axis perpendicular to the rod and 6 in. in from the free end of the rod.

461. A solid cast-iron pulley has a cross-section as shown in Fig. 396. Determine the moment of inertia of the pulley with respect to an axis through the center of the shaft. *Ans. 9.87 slugs-ft².*

462. Determine the moment of inertia of the cast-iron flywheel shown in Fig. 397. The wheel has six spokes of elliptic section as indicated.

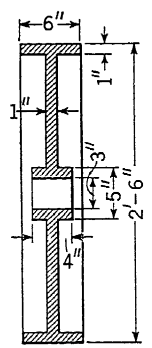


FIG. 396

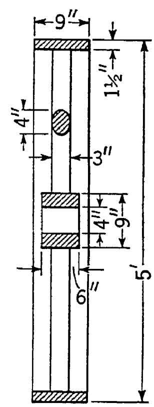


FIG. 397

463. Determine the moment of inertia of the frustum of the steel cone shown in Fig. 398 with respect to an axis parallel to its geometric axis and tangent to the base circle.

464. Fig. 399 represents an ordinary fly-ball governor. What is its moment of inertia with respect to the vertical axis of rotation?

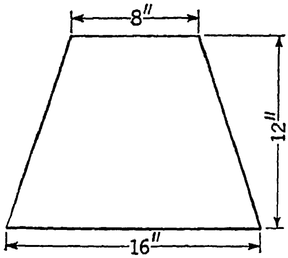


FIG. 398

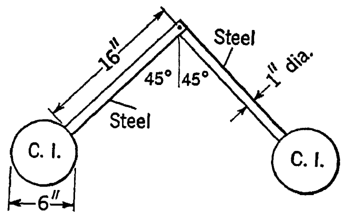


FIG. 399

465. A steel sphere 12 in. in diameter has a hole 2 in. in diameter passing through its center as shown in Fig. 400. What is its moment of inertia with respect to the Y axis? *Ans. 0.77 slugs-ft²*.

466. Compute the moment of inertia of the sphere of Problem 465 with respect to a tangent parallel to the Y axis.

467. Calculate the moment of inertia of a right circular cone with height h and radius of base r, with respect to a diameter of the base.

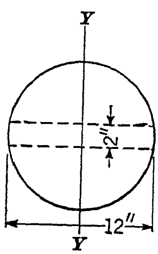


FIG. 400

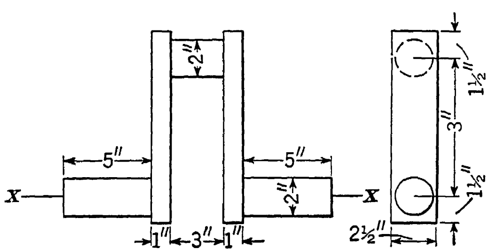


FIG. 401

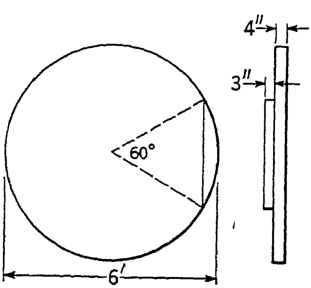


FIG. 402

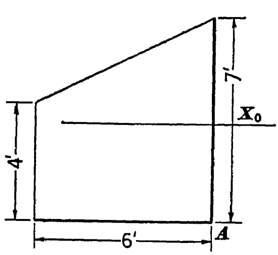


FIG. 403

468. Fig. 401 represents a single-throw steel crankshaft. What is its moment of inertia with respect to the X axis?

469. Solve for the moment of inertia of the steel locomotive drive-wheel shown in Fig. 402 with respect to the axis of rotation. Assume that the wheel is a disk 4 in. thick and that the balance weight is a segment 3 in. thick which is attached to the side of the disk. *Ans. 670 slugs-ft²*.

470. The thin plate in Fig. 403 weighs 1 lb per sq ft. Compute its moment of inertia with respect to the X_0 axis.

471. For the plate of Problem 470 determine the polar moment of inertia for an axis normal to the plate through corner A.

472. The 120° sector of a circular steel disk in Fig. 404 is 4 in. thick. Compute its moment of inertia with respect to an axis through point O normal to the plane of the disk.

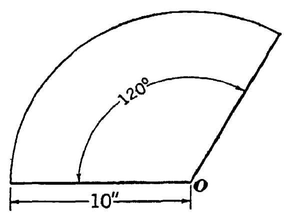


FIG. 404

CHAPTER 13

KINEMATICS OF A PARTICLE

112. Introductory Statement.—Kinematics is the science which expresses the mathematical relationships existing between displacement, velocity, acceleration, and time. It deals with the motion of a particle or a geometric form without regard to the forces which cause or affect the motion. The term geometric form is here taken to mean any rigid form or shape which is considered to be devoid of all physical properties such as weight or mass. Kinematics is therefore simply the study of motion in the abstract.

From the standpoint of Mechanics, a particle is a material point or a quantity of matter so small that it can be thought of as having no dimensions. Any physical body consists of a group of particles joined together in a definite form or relationship one to the other.

Often the dimensions of a physical body are very small as compared to its range of motion. Stars and projectiles come under this classification and can be treated as particles. Likewise, when a physical body moves along a straight-line path in such a manner that all particles of the body move along parallel straight lines, then the body may be considered as moving with the motion of a material particle.

113. Motion of a Particle.—A particle can move in two ways only:

(*a*) If the particle moves along a straight-line path, the motion is called rectilinear.

(*b*) If the particle moves along a curved path, the motion is known as curvilinear.

The discussions in this text will be limited to rectilinear motion and to plane curvilinear motion, that is, motion along a curved path which lies in a single plane.

114. Linear Displacement.—*The linear displacement of a particle is its change of position with reference to some fixed point.* The linear displacement is independent of the path traveled in moving from the original position to the final position.

If a particle travels along any path, such as from A to B to C in Fig. 405, its linear displacement from A is represented by the vector AC. The displacement is a vector quantity, or a directed distance, since the line AC has magnitude, direction, and position. Any convenient units of length, such as feet or inches, may be used to express the displacement.

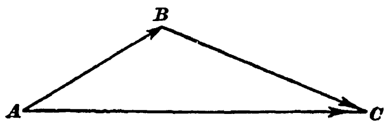


FIG. 405

PROBLEMS

473. A material point or particle moves northeast for 2 miles and then east for 3 miles. Determine the amount and direction of its displacement from the starting point.

474. A particle A moves 500 ft east from a given point while a particle B is moving 800 ft south 30° east. Determine the amount and direction of the displacement of particle A from particle B.

475. A toy balloon ascends 5,000 ft while traveling 2 miles west from its starting point. Compute the magnitude of its displacement.

115. Linear Speed and Velocity.—*Linear speed is the time rate of travel.* If a particle travels 30 ft per sec, it has a speed of 30 ft per sec. Linear velocity is the rate of travel in a definite direction. *Velocity is the time rate of displacement.* Since displacement is a vector quantity, velocity must also be a vector quantity. Velocity therefore has magnitude, direction, and position. A particle traveling 30 ft per sec along some definite straight line has a velocity of 30 ft per sec.

If a particle moves along a straight line until its displacement is Δs during the time Δt, then its average velocity during the time Δt is $v_{\text{avg}}=\dfrac{\Delta s}{\Delta t}$. During this period the velocity may have varied from zero to a maximum value and back again to zero. As the increments of distance and time become smaller and smaller, and as Δt approaches zero as a limit, the instantaneous velocity becomes

$$v=\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}=\frac{ds}{dt}$$

If the particle travels equal distances in equal periods of time, the velocity is uniform and is found from the relation $v=\dfrac{s}{t}$, where

s is the total displacement which occurs during the time interval t. Linear velocity is generally expressed in feet per second or miles per hour.

EXAMPLE

If an object is dropped from the top of a building and its motion is described by the equation $s=ct^2$, where the constant $c=14$, what is the speed of fall 4 seconds after release?

$$v=\frac{ds}{dt}$$

By differentiation,

$$\frac{ds}{dt}=2\times 14\,t=28\,t=28\times 4=112 \text{ ft per sec}$$

PROBLEMS

476. A particle moves along a straight line at a constant velocity of 5 ft per sec. What is its displacement after 5 seconds? *Ans. 25 ft*

477. If the particle in Problem 476 starts from rest, what average velocity would be required to produce the same displacement in 10 seconds? What maximum velocity will the particle attain?

478. An automobile traveled 30 miles per hour for 10 minutes, 45 miles per hour for 5 minutes, and 60 miles per hour for 15 minutes. What total distance did the car travel? What was the average speed of the car in miles per hour?

479. A body is dropped from a building and its motion is described by the equation $s=\frac{1}{2}gt^2$, where g has the value 32.2 ft per sec². What is the speed of the body after 5 seconds? How far will the body fall during the fifth second?

116. Linear Acceleration.—If the linear velocity or rate of displacement is variable, the particle is said to be traveling with non-uniform velocity. *The time rate of change of the linear velocity is the linear acceleration.*

If a change in linear velocity Δv occurs during a time interval Δt, the average linear acceleration is $a_{\text{avg}}=\frac{\Delta v}{\Delta t}$. As the linear velocity and the time increments become smaller and smaller, and as Δt approaches zero as a limit, the linear acceleration is

$$a=\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}=\frac{dv}{dt}$$

Since $v=\frac{ds}{dt}$,

$$\frac{dv}{dt}=\frac{d^2s}{dt^2} \text{ and } a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

Velocity is a vector quantity; therefore, acceleration is also a vector quantity.

The most commonly used units for acceleration are feet per second per second or feet per second².

If dt is eliminated from the equations $v=\frac{ds}{dt}$ and $a=\frac{dv}{dt}$, we obtain the equation $v\,dv=a\,ds$.

If the relationships of the variables are known, *the basic differential equations which may be used to solve any rectilinear kinematic problem are*

$$v=\frac{ds}{dt};\ a=\frac{dv}{dt}=\frac{d^2s}{dt^2};\ \text{and } v\,dv=a\,ds$$

EXAMPLE

Calculate the acceleration of the object in the Example of Art. 115.

$$v=\frac{ds}{dt}=2\times 14\,t=28\,t$$

$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=28 \text{ ft per sec}^2$$

Therefore, the object falls with a constant acceleration as long as its motion remains within the limitations of the equation $s=ct^2$.

PROBLEMS

480. A particle attains a velocity of 20 ft per sec after traveling 50 ft. What constant linear acceleration did it receive? *Ans. 4 ft per sec².*

481. If the particle in Problem 480 moves an additional 60 ft in 2 sec, what is its velocity after traveling the 110 ft? What constant acceleration, acting during the entire distance, would produce the same final velocity?

482. If the speed of an automobile is changed from 20 mi per hr to 60 mi per hr in 40 sec, what is the acceleration in ft per sec per sec?

483. A particle moves on a straight-line path in such a manner that its displacement, at any instant, from a fixed point O on its path is given by the equation $s=t^3+3t^2+8t$. What displacement, velocity, and acceleration will the particle attain after 10 sec?

484. How far will the particle in Problem 483 travel during the tenth second? *Ans. 336 ft.*

485. What constant acceleration acting for 10 sec would produce the final velocity attained in Problem 483?

486. A particle falling freely in a vacuum moves so that its displacement at any instant is given by $s=16.1\,t^2$. What acceleration does the particle receive? What are its displacement and its velocity after falling 5 sec?

117. Fundamental Equations for Rectilinear Motion of a Particle With Uniform Acceleration.—When the law which governs the acceleration of a particle is known and the motion of the particle follows the given law, the general differential equations of rectilinear motion $v=\frac{ds}{dt}$, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$, and $v\,dv=a\,ds$ may be used to derive special equations which express the relationships existing between s, v, a, and t.

The most frequently encountered law is that expressed by $a=C$, where C is a constant, as in the case of a freely falling body where $a=g=C$ and the acceleration g is a constant throughout the motion. A freely falling body is one which falls under the influence of gravity alone, air resistance being neglected. *Formulas for rectilinear motion with constant acceleration* may be derived by integrating between definite limits or by general integration and evaluation of the constants of integration. The latter method will be used here because it is a technique which the student should become familiar with.

$$\int dv=\int a\,dt$$
$$v=a\,t+C_1$$

When $t=0$, the velocity $=v_0$. Hence, $C_1=v_0$.

$$v=v_0+a\,t \tag{1}$$

Since $v=\frac{ds}{dt}$,

$$\int ds=\int v_0\,dt+\int a\,t\,dt$$
$$s=v_0\,t+\frac{1}{2}\,a\,t^2+C_2$$

When $t=0$, the displacement $s=0$. Therefore, $C_2=0$.

$$s=v_0\,t+\frac{1}{2}\,a\,t^2 \tag{2}$$

$$\int v\,dv=\int a\,ds$$
$$\frac{v^2}{2}=a\,s+C_3$$

When $s=0$, $v=v_0$. Therefore, $C_3=\frac{v_0^2}{2}$.

$$v^2=v_0^2+2\,a\,s \tag{3}$$

Eliminating a from equations (1) and (2) gives

$$s=\left(\frac{v+v_0}{2}\right)t \tag{4}$$

Equations (1), (2), (3), and (4) are true only when the acceleration a is constant.

When using these equations, the student must be careful to be consistent in the use of signs. *Usually the direction of the initial motion is taken as the positive direction.* If the quantities s, v, and a have the same direction as the initial motion, they are given the positive sign. If their direction is opposite to that of the initial motion, they are given the negative sign.

EXAMPLE 1

An automobile has a speed of 30 mi per hr when the brakes are applied. The car is slowed down at the rate of 8 ft per sec per sec. What time will be required to stop the car and how far will it travel while stopping?

30 miles per hr $=$ 44 ft per sec

$$v=v_0+at$$
$$0=44+(-8)\,t$$
$$t=5.5 \text{ sec}$$
$$s=v_0t+\tfrac{1}{2}at^2$$
$$s=44\times5.5+\frac{1}{2}(-8)\,(5.5)^2=121 \text{ ft}$$

Another solution follows:

$$\int dv=\int a\,dt$$
$$\int_{44}^{0} dv=\int_{0}^{t}(-8)\,dt$$
$$t=5.5 \text{ sec}$$
$$\int v\,dv=\int a\,ds$$
$$\int_{44}^{0} v\,dv=\int_{0}^{s}(-8)\,ds$$
$$s=121 \text{ ft}$$

EXAMPLE 2

A material particle passes a given point while traveling to the right with a velocity of 100 ft per sec. The particle is receiving an acceleration of 25 ft per sec per sec acting to the left. What are the displacement and the velocity of the particle 10 sec after passing the given point?

$$v=v_0+at$$
$$v=100+(-25)\ 10$$
$$v=-150 \text{ ft per sec} \leftarrow$$
$$s=v_0t+\frac{1}{2}\,at^2$$
$$s=100\times 10+\frac{1}{2}(-25)\ 100$$
$$s=-250 \text{ ft} \leftarrow$$

Another solution follows:

$$\int dv=\int a\,dt$$
$$\int_{100}^{v} dv=\int_{0}^{10} (-25)\,dt$$
$$v=-150 \text{ ft per sec} \leftarrow$$
$$\int v\,dv=\int a\,ds$$
$$\int_{100}^{-150} v\,dv=\int_{0}^{s} (-25)\,ds$$
$$s=-250 \text{ ft} \leftarrow$$

The displacement s was given the positive sign in the above equation. This implied that the final displacement was in the direction of the initial motion, or to the right of the given point. When the equation was solved, s was found to have a negative value. This negative sign indicates that the original assumption as to the displacement of the particle was incorrect. The final position of the particle was to the left of the given point.

PROBLEMS

487. An automobile traveling 60 mi per hr is brought to rest in 3 min. Find the constant acceleration required and the distance traveled by the car while it is coming to rest. *Ans. 0.488 ft per sec²; 7,920 ft.*

488. A particle moving with a velocity of 10 ft per sec is given an acceleration of 2 ft per sec per sec. What are its velocity and its displacement after 40 sec?

489. An elevator is ascending at the rate of 480 ft per min at the instant it passes a given point. The elevator then receives a constant downward acceleration of 1 ft per sec per sec. What velocity will the elevator have after 30 sec? What is its displacement?

490. Two elevators operating in parallel shafts approach each other from positions which are 500 ft apart. The upper car has a downward acceleration of 1 ft per sec per sec, and the lower car is being accelerated upward 2 ft per sec per sec. When and where will they pass if the lower car starts 1 sec after the upper car?

491. Derive formulas (1), (2), and (3), Art. 117, by integration between definite limits.

492. Automobile A, which receives a constant acceleration of $\frac{1}{2}$ ft per sec per sec, starts from a given point 30 sec before a second car B passes the same point at 30 mi per hr. If the car B is decelerating at the rate of 1 ft per sec per sec, at what distance from the given point will the cars pass? Explain the answer.

493. Two cars approach each other on a straight road from points 1,000 ft apart. The car A has an initial velocity of 60 mi per hr and is being decelerated at the rate of 2 ft per sec per sec. Car B has an initial velocity of 15 mi per hr and is accelerating at the rate of 1.2 ft per sec per sec. When will the cars meet, and how far will car A have traveled? *Ans. 9.4 sec; 738.8 ft.*

118. Freely Falling Particles.—A freely falling particle is here considered to mean any particle which falls under the influence of gravity alone, the resistance of the air being neglected.

The acceleration due to gravity varies for different locations, according to the following equation:

$$g = 32.089\ (1 + 0.00524 \sin^2 \theta)\ (1 - 0.000000096h)$$

where θ is the latitude in degrees and h is the elevation above the sea level in feet. The value of g decreases as the elevation above sea level increases. It will be found that the most extreme variations of latitude and altitude which are encountered in ordinary engineering calculations produce a variation of less than 1 per cent in the value of g. Therefore, for most practical calculations it is sufficiently accurate to assume that g has a constant value of 32.2. If this assumption is made, freely falling particles will follow the relationships established by the equations of Art. 117, in which g is substituted for a. Thus,

$$v = v_0 + g\,t \tag{1}$$

$$s = v_0\,t + \frac{1}{2}\,g\,t^2 \tag{2}$$

$$v^2 = v_0^2 + 2\,g\,s \tag{3}$$

$$s = \left(\frac{v + v_0}{2}\right) t \tag{4}$$

EXAMPLE

A small ball is thrown upward with an initial velocity of 50 ft per sec from the top of a 100-ft building. At the same instant, another is thrown upward from the ground with an initial velocity of 100 ft per sec. Where, and how long after starting, will they pass?

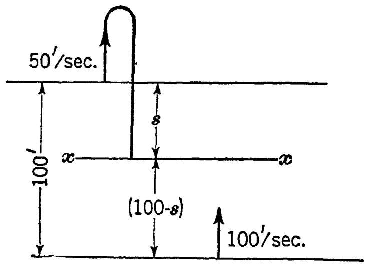


FIG. 406

In Fig. 406, assume that the balls pass at the level xx. The equation for the first ball will then be

$$-s = 50\,t - \frac{1}{2} \times 32.2\,t^2 \tag{1}$$

For the second ball,

$$(100 - s) = 100\,t - \frac{1}{2} \times 32.2\,t^2 \tag{2}$$

$$\begin{array}{rcll} 100 - s & = & 100\,t - 16.1\,t^2 & (2) \\ -s & = & 50\,t - 16.1\,t^2 & (1) \\ + \quad - & & + & \\ \hline 100 & = & 50\,t & \\ & & t = 2 \text{ sec} & \end{array}$$

Substituting in equation (1) gives

$$-s = 50(2) - 16.1(4)$$
$$s = -35.6 \text{ ft}$$

The negative sign indicates that the location assumed for the meeting place was incorrect. The balls pass at an elevation 35.6 ft above the top of the building.

PROBLEMS

494. A ball falls from the top of a 200-ft building. How long will it take to reach the ground, and with what velocity will it strike? *Ans. 3.52 sec; 113.5 ft per sec.*

495. If the ball of Problem 494 is thrown upward with an initial velocity of 100 ft per sec, how high above the top of the building will it go? How long will it take to attain its maximum height? When will it reach the ground?

496. A ball is dropped from the top of a 400-ft building at the same instant at which another is thrown upward from the ground. The balls pass at a point 125 ft from the top of the building. What was the initial velocity of the second ball?

497. If in Problem 496 the lower ball is thrown upward with a starting velocity of 125 ft per sec 2 sec after the upper ball is dropped, when and where will the balls pass?

498. A body slides down a smooth plane inclined 30° with the horizontal. If the initial velocity of the body is 10 ft per sec and the plane is 50 ft long, what velocity will be attained at the end of the incline? What time will elapse?

499. At an elevation of 300 ft from the ground, a small ball is dropped from a balloon which, at the instant the ball is released, is ascending with a velocity of 480 ft per min and is being accelerated upward 3 ft per sec per sec. When and with what velocity will the ball strike the ground? *Ans. 4.56 sec; 139.2 ft per sec.*

119. Relative Motion.—Quite often when two particles are in motion with respect to the earth, it is desirable to study their motion, displacement, velocity, or acceleration relative to each other. Such quantities are known as *relative* motion, displacement, velocity, and acceleration.

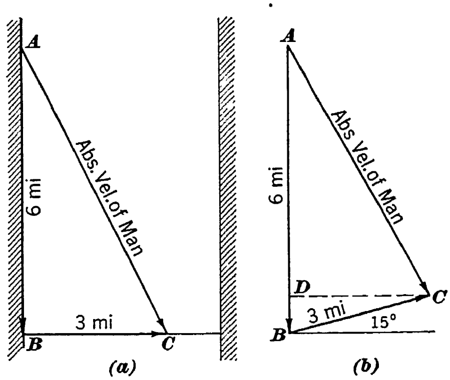


FIG. 407

All motion is relative; however, custom has decided that *motion relative to the earth's surface or some fixed point on the surface shall be designated as absolute.*

Let Fig. 407 (a) represent a river flowing due south at 6 miles per hour. If a man who can swim 3 miles per hour in still water

heads straight out from the river bank, what is his absolute velocity, or his velocity with respect to the river bottom?

In Fig. 407 (*a*), the vector BC represents the man's velocity relative to the water. Vector AB is the absolute velocity of the water, or the velocity of the water relative to the river bottom. Vector AC is the velocity of the man relative to the river bottom, or the absolute velocity of the man. Thus, the absolute velocity of the man is the vector sum of the absolute velocity of the river and the velocity of the man relative to the water.

The vectors of Fig. 407 (*a*) could also be used to represent displacements or accelerations of the respective particles.

The discussion just given leads to the following general theorem for relative motion:

The absolute displacement, velocity, or acceleration of any particle A is equal to the vector sum of the absolute displacement, velocity, or acceleration of another particle B and the displacement, velocity, or acceleration of particle A relative to particle B.

Absolute velocity of A = Absolute velocity of B ⇸ Relative velocity of A to B (vector sum).

$$v_A = v_B \nrightarrow v_{\frac{A}{B}}$$

EXAMPLE 1

If the river in the preceding discussion is 1 mi wide and the man heads up stream 15°, where will he reach the opposite bank? How long will it take?

In Fig. 407 (*b*), BC is the velocity of the man relative to the water; AB is the absolute velocity of the water; and AC is the absolute velocity of the man with respect to the river bed. The magnitude and direction of the vector AC may be determined graphically or analytically.

The component of the absolute velocity across the river is

$$CD = 3 \times 0.966 = 2.898 \text{ mi per hr}$$

The time required to cross the river is

$$\frac{1}{2.898} = 0.346 \text{ hr}$$

$$BD = 3 \times 0.259 = 0.777 \text{ mi per hr}$$

The component of the absolute velocity down the river is

$$AD = AB - BD = 5.223 \text{ mi per hr}$$

The distance below A is

$$5.22 \times 0.346 = 1.805 \text{ mi}$$

EXAMPLE 2

An airplane can fly 90 mi per hr in still air. If the wind is blowing 30 mi per hr from the southeast, in what direction should the plane be headed if it is desired to fly due north? How long will it take to travel 200 mi?

In this problem the absolute velocity of the wind is given in amount and direction. The direction of the absolute velocity of the plane is due north; its amount is not known. The velocity of the plane relative to the wind is 90 mi per hr, but its direction is unknown.

In Fig. 408 lay down vector AB to represent the absolute velocity of the wind. Draw through A a line of indefinite length, running due north. This line represents the direction of the absolute velocity of the plane. From B draw vector BC, with a magnitude of 90, so that it will close the triangle ABC. Then AC represents the magnitude and direction of the absolute velocity of the plane; and BC represents the velocity of the plane relative to the air and gives the direction in which the plane must be headed if it is to fly due north.

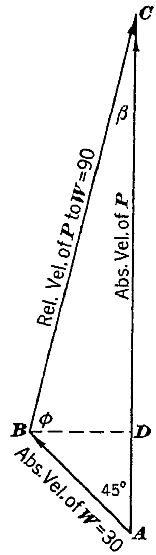


FIG. 408

$$BD = 30 \times 0.707 = 21.21 \text{ mi}$$

$$\cos \phi = \frac{21.21}{90};\ \phi = 76.4°;\ \beta = 13.6°$$

The absolute velocity of the plane is

$$21.21 + 90 \sin 76.4° = 108.6 \text{ mi per hr}$$

The time required to fly 200 mi is

$$\frac{200}{108.6} = 1.84 \text{ hr} = 1 \text{ hr } 50.4 \text{ min}$$

PROBLEMS

500. Two trains are approaching each other on parallel tracks. Train A is 400 ft long and is traveling at 60 mi per hr. Train B is 100 ft long and is traveling 15 mi per hr. How long will it take the trains to completely pass each other? *Ans. 4.54 sec.*

501. Two automobiles move away from the intersection of two roads which make an angle of 60° with each other. Car A receives an acceleration of 0.9 ft per sec per sec, and car B an acceleration of 0.75 ft per sec per sec. Determine the relative displacement, velocity, and acceleration of the two cars 20 sec after leaving the intersection.

502. If the two trains of Problem 500 are traveling on tracks which intersect at 120° and both trains are approaching the intersection, what is their relative velocity?

503. The weather vane on a ship points southeast when the ship is moving east at the rate of 20 mi per hr. If the velocity of the wind is 30 mi per hr, from what direction is it blowing?

504. The mechanism in Fig. 409 consists of a crank OA with a sliding block A attached. The block slides along the rocker arm BC as the crank OA rotates about O at a constant angular velocity of 40 rpm. Find the absolute velocity v_D of the point D on the rocker arm BC, which coincides with the point A on the block, and the velocity $v_{\frac{A}{D}}$ of the block A relative to the rocker arm BC.

505. The water enters the inward flow turbine in Fig. 410 with a velocity of 150 ft per sec and at an angle of 30° with the radius extended. The tangential rim velocity of the turbine blades is 60 ft per sec. Determine the angle of the outer blade surface of the turbine, if the water enters the turbine along a tangent to the blade surface.

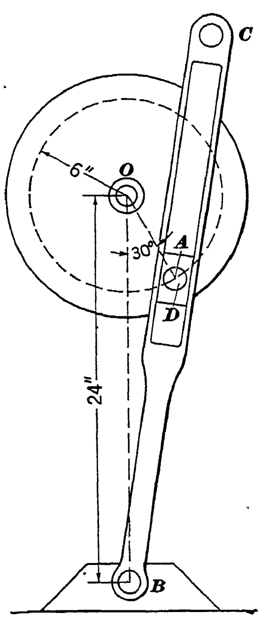


FIG. 409

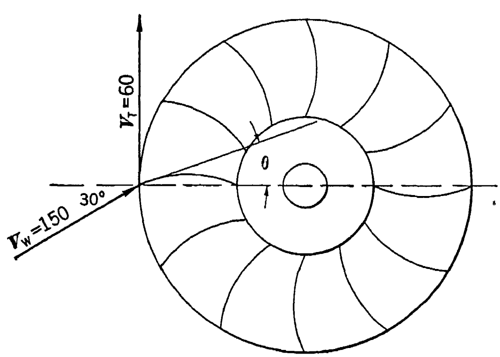


FIG. 410

120. Rectilinear Motion of a Particle With Variable Acceleration.—As stated in Art. 117, the equations $v=\frac{ds}{dt}$, $a=\frac{dv}{dt}$, and $v\,dv=a\,ds$ must be used when the acceleration is variable.

EXAMPLE

A particle moves with an acceleration $a=s$. It has an initial velocity of 6 ft per sec. What is the velocity after the particle moves 10 ft? What is the elapsed time?

$$\int_6^v v\,dv=\int_0^{10} a\,ds=\int_0^{10} s\,ds$$

$$v=11.66 \text{ ft per sec}$$

$$\int v\,dv=\int s\,ds$$

$$\frac{v^2}{2}=\frac{s^2}{2}+C$$

When $s=0$, $v=6$. Hence, $C=18$.

$$v^2=s^2+36$$

$$v=\sqrt{s^2+36}=\frac{ds}{dt}$$

$$\int_0^t dt=\int_0^{10}\frac{ds}{\sqrt{s^2+36}}.$$

$$t=\log_e\left(s+\sqrt{s^2+36}\right)\Big]_0^{10}$$

$$t=3.08 \text{ sec}$$

PROBLEMS

506. A point moves with rectilinear motion so that its acceleration is $a=-ks$, where s is the distance from the starting point. When $s=3$ ft, the velocity is 4 ft per sec; when $s=5$ ft, the velocity is 3 ft per sec. What is s when $v=0$? 6.08

507. A particle moving in a straight line has an acceleration $a=3t-12$. Its initial velocity is 12 ft per sec. (*a*) What is its velocity after 8 sec? (*b*) What is its displacement after 10 sec?

508. A particle moving in a straight line with an initial velocity of 5 ft per sec is subjected to an acceleration $a=4-2t$. (*a*) When will the velocity be zero? (*b*) How far will the particle travel in 10 sec?

121. Displacement and Velocity Along a Curved Path.—In Fig. 411, $ABCD$ is any plane curved path followed by a particle.

The displacement of the particle from A, when at B, is given by the vector AB; and, similarly, the displacement for point C is given by the vector AC. The displacement in curvilinear motion is independent of the path traveled and depends only on the vector distance between the starting and final positions.

As a particle moves along any plane curve from A to C, Fig. 411, a distance s during the time t, the average speed along the curve is given by $\frac{s}{t}$. If the motion along the curve is such that equal distances are traveled in equal periods of time, then the motion is known as uniform curvilinear motion. If the particle travels unequal distances in equal periods of time, the motion is non-uniform curvilinear motion.

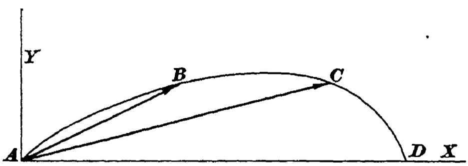


Fig. 411

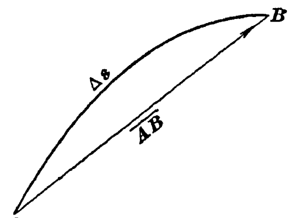


Fig. 412

If a particle moves from A to B, Fig. 412, a distance Δs along the curve in the time Δt, then the average speed over the distance Δs is $\frac{\Delta s}{\Delta t}$. As Δs and Δt are allowed to approach zero as a limit, the ratio $\frac{\Delta s}{\Delta t}$ approaches the instantaneous speed at the point A.

$$v_A = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

The ratio $\frac{\overline{AB}}{\Delta t}$, where $\overline{AB}$ represents the length of the chord AB or the linear displacement of B from A, is the average velocity between A and B, because $\overline{AB}$ has definite magnitude, direction, and position. As the interval of time Δt is allowed to approach zero as a limit, the chord $\overline{AB}$ approaches the curve Δs; and, in the limit, $\overline{AB}$ will be tangent to the curve at A. The direction of the instantaneous velocity for any point on the curve is thus established as along the tangent to the curve at the given point. The magnitude of the instantaneous velocity is the same as the speed along the curve at the particular point.

122. Acceleration During Plane Curvilinear Motion.—According to definition, acceleration is the time rate of change of velocity. Since velocity is a vector quantity, and thus has both magnitude and direction, any change in either the magnitude or the direction of the velocity of a particle indicates that the particle has received an acceleration.

When a particle moves along any plane curve, the direction of its velocity is constantly changing. The magnitude of the velocity, or the speed of the particle along the curve, may or may not change. It therefore follows that any motion along a plane curve always involves a change in velocity and thus requires an acceleration.

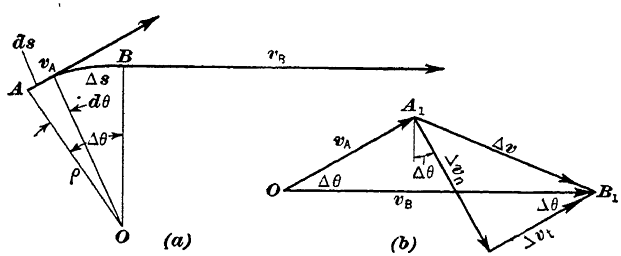


FIG. 413

Fig. 413 (*a*) represents any plane curve; and v_A and v_B are the instantaneous velocities of a particle moving along the curve from A to B, when it is at points A and B. In Fig. 413 (*b*), the velocity vectors v_A and v_B are drawn to scale and equal and parallel to the velocities at points A and B. The vector Δv represents the total change in velocity which occurs between A and B. If the time required to pass along the curve from A to B is Δt, then $\frac{\Delta v}{\Delta t}$ is the average rate at which the velocity is changing between the two points, or is the average acceleration.

It will be observed that the vector Δv is not parallel to the velocity at either A or B, and therefore the acceleration $\frac{\Delta v}{\Delta t}$ is not parallel to the direction of v_A or v_B. Generally it is more convenient to resolve the acceleration into two components, one in the direction of the tangent and the other normal to the tangent or along the radius vector to the center of curvature.

123. Normal and Tangential Components of Curvilinear Acceleration.—A particle moving along the curved path in Fig.

413 (*a*) from A to B travels a distance Δs during the time Δt. The center of curvature for the portion of the curve at A is O, and the radius of curvature is ρ. The velocity at A is v_A and that at B is v_B.

In Fig. 413 (*b*) the vector Δv may now be resolved into the components Δv_t parallel to the velocity v_A and Δv_n normal to v_A and parallel to the radius of curvature ρ. The average tangential and normal components of the acceleration are then given by the following:

$$a_t=\frac{\Delta v_t}{\Delta t} \text{ and } a_n=\frac{\Delta v_n}{\Delta t}$$

As Δt and $\Delta\theta$, Fig. 413 (*b*), approach zero as a limit, the component Δv_t will approach the direction of OB_1, and the limiting value of $\cos\Delta\theta$ as $\Delta\theta$ approaches zero is unity. Therefore,

$$a_t=\lim_{\Delta t\to 0}\frac{\Delta v_t}{\Delta t}=\frac{v_B-v_A\cos\Delta\theta}{dt}=\frac{v_B-v_A}{dt}=\frac{dv}{dt} \quad . \quad (1)$$

From this relation, we see that in the limit the tangential component Δv_t of the change in velocity coincides with the direction OB_1 and has a magnitude equal to the change in speed dv which occurs during the differential time interval dt.

As Δt and $\Delta\theta$ approach zero as a limit, they become dt and $d\theta$, and Δv_n approaches $v_A \sin d\theta$. Thus,

$$a_n=\lim_{\Delta t\to 0}\frac{\Delta v_n}{\Delta t}=\frac{v_A\sin d\theta}{dt}=\frac{v\,d\theta}{dt} \quad .$$

But, from Fig. 413 (*a*), $d\theta=\frac{ds}{\rho}$; and, by definition, $\frac{ds}{dt}=v$. Hence,

$$a_n=\frac{v}{dt}\,\frac{ds}{\rho}=\frac{v^2}{\rho} \quad (2)$$

Equation (2) shows an important difference between rectilinear motion and motion along any plane curve. If a particle moves along a straight line, it may or may not have an acceleration. If it does have an acceleration, the acceleration must be along the line of motion.

If a particle moves along any plane curve, equation (2) shows that the particle *must receive an acceleration toward the center of*

curvature if there is to be any motion. There may or may not be an acceleration in the tangential direction.

PROBLEMS

509. A particle travels around a circle of 10-ft radius 4 times in one minute. What is the acceleration normal to the curve? *Ans. 1.75 ft per sec*^{2}.

510. A pulley 3 ft in diameter attains a speed of 200 rpm in 40 sec. What are the tangential and normal accelerations of a particle on the rim of the pulley 20 sec after the pulley starts?

511. A particle moves along a curved path at a constant speed of 30 ft per sec. If the radius of curvature changes from 40 ft to 60 ft while the particle is traveling from A to B in 10 sec, what is the normal acceleration from A to B?

124. Motion of a Projectile, Air Resistance Neglected.—For the purpose of the following general discussion, a projectile is any body which, having received an initial velocity, then moves under the influence of gravity alone. The flight of the real projectile is influenced by such factors as the size, shape, and rotative speed of the projectile and the condition of the air through which the projectile is passing. Since it is impossible to consider all these factors in a text of this character, the analysis which follows will assume that the projectile is a material particle traveling in a vacuum and influenced only by the acceleration due to the force of gravity.

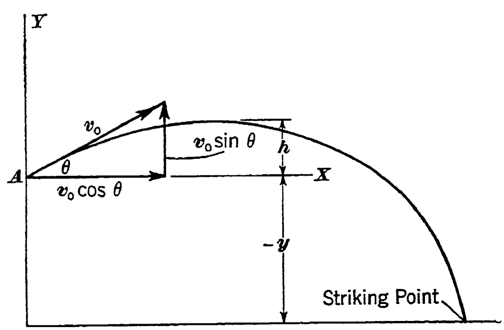


FIG. 414

Assume that a projectile starts from A, Fig. 414, with an initial velocity v_0. Resolve v_0 into its horizontal and vertical components.

$$v_h = v_0 \cos \theta \text{ and } v_v = v_0 \sin \theta$$

Since the acceleration of gravity acts in the vertical direction, it influences the vertical component of the velocity only. *The*

horizontal component $v_h = v_0 \cos\theta$ remains unchanged throughout the flight of the projectile. The vertical component of the velocity obeys the laws governing freely falling bodies, as stated in Art. 118.

If y represents the difference in elevation between the starting point and the striking point of the projectile, the total time of flight t can be determined from the following equation:

$$-y = t\, v_0 \sin\theta - \frac{1}{2}\, g\, t^2 \tag{1}$$

Consistency in the use of signs, as discussed in Art. 117, is necessary. In Fig. 414 the striking point is below the starting point or in the opposite direction from the vertical component of the initial motion; therefore, the quantities y and g are given the negative sign.

The horizontal distance traveled by the projectile is:

$$x = t\, v_0 \cos\theta \tag{2}$$

If t is eliminated from equations (1) and (2), the equation of the path of flight will be obtained. Thus,

$$-y = x \tan\theta - \frac{g\, x^2}{2v_0^2 \cos^2\theta}$$

This is the equation of a parabola.

EXAMPLE 1

A rifle is fired from the top of a 300-ft building. The initial velocity of the bullet is 1,200 ft per sec, and the rifle is pointed 15° above the horizontal. When and where will the bullet strike the ground? Determine the maximum height attained by the bullet and the time required to reach that height.

$$-300 = 1{,}200 \times 0.259\, t - \frac{1}{2} \times 32.2\, t^2$$

$$t^2 - 19.3\, t + \left(\frac{19.3}{2}\right)^2 = 18.65 + \left(\frac{19.3}{2}\right)^2 = 111.65$$

$$t - 9.65 = \pm 10.55$$
$$t = -0.9 \text{ or } 20.2 \text{ sec}$$

Hence, the total time of flight is 20.2 sec, and the distance from the building to the striking point is

$$x=20.2\times 1{,}200\times 0.966=23{,}415.8 \text{ ft}$$

When the projectile reaches its maximum height, the vertical component of its velocity is zero.

$$v=v_0-g\,t$$
$$0=1{,}200\times 0.259-32.2\,t$$
$$t=9.65 \text{ sec}$$

The bullet therefore attains its maximum height 9.65 sec after starting its flight.

$$v^2=v_0^2+2\,g\,h$$

$$0=(1{,}200\times 0.259)^2+2(-32.2)\,y$$
$$y=1{,}505 \text{ ft}$$

The maximum height is

$$1{,}505+300=1{,}805 \text{ ft}$$

EXAMPLE 2

A projectile has a muzzle velocity of 1,000 ft per sec. What angle of elevation must the gun have, if the projectile is to hit a target 2,000 ft away and 500 ft above the gun?

$$v_h=1{,}000 \cos\theta \text{ and } v_v=1{,}000 \sin\theta$$
$$500=1{,}000 \sin\theta\, t-\frac{1}{2}\times 32.2\times t^2$$
$$2{,}000=1{,}000 \cos\theta\, t$$

By eliminating t from these equations, it is found that

$$\theta=16.1^\circ \text{ with the horizontal}$$

PROBLEMS

512. If a stone is thrown horizontally with a velocity of 20 ft per sec from the top of a cliff 150 ft high, how far from the face of the cliff will it strike? If sound travels approximately 1,080 ft per sec, how long after the stone is thrown will the sound of the impact be heard? *Ans. 61 ft; 3.2 sec.*

513. Derive the formula for the range of a projectile. Range is the horizontal distance of flight from the firing point. What angle of elevation of the gun will theoretically produce the greatest range?

514. A bombing plane in level flight at 25,000 ft is traveling at 300 mi per hr. How far ahead of the target horizontally should the bomb be released? What is the time of flight of the bomb?

515. A projectile is discharged from a gun elevated 15° above the horizontal. It strikes the top of a 600-ft building 6,000 ft away. What were the muzzle velocity, the maximum height attained, and the total time of flight?

516. A rifle with a muzzle velocity of 1,100 ft per sec is fired from the top of a building 400 ft high. The gun is pointed 15° below the horizontal. When and where will the bullet hit the ground?

517. A gun with a muzzle velocity of 1,000 ft per sec is located on a hill 500 ft high. What angle of elevation should the gun have if the projectile is to strike a ship 10,000 ft from the gun?

518. A body slides down a smooth plane, inclined 30° with the horizontal. If the plane is 20 ft long and its lower end is 50 ft from the ground, where will the body strike the ground, and what is the total time required? *Ans. 31 ft; 2.987 sec.*

125. Graphical Relation Between Linear Displacement, Speed, Acceleration, and Time.—The discussion which follows is limited to the motion of a particle along a straight line.

There are many cases of straight-line motion for which it is difficult to write equations descriptive of the motion. It is, however, generally possible to obtain sufficient data for plotting curves which describe the motion.

Displacement-Time Curve.—If the linear displacements of a particle for several periods of time can be obtained, the displacement-time curve can be plotted. Such a curve is shown in Fig. 415 (*a*). The ordinates of the curve are the linear displacements of a given particle for definite periods of time. If a tangent is drawn to such a curve at any point A, the slope of the tangent $\frac{ds}{dt}$ will be the instantaneous speed of the particle when the displacement is that corresponding to point A. This slope can be determined graphically by the construction shown at A, Fig. 415 (*a*).

$$\text{Slope} = \frac{ds}{dt} = \frac{\text{feet}}{\text{seconds}} = v$$

Speed-Time Curve.—The slope of the displacement-time curve (instantaneous value of the speed) can be determined for a number of points along the curve. These values of the slope can then be

plotted as ordinates against the corresponding values of the elapsed time as abscissae, to give a second curve known as the speed-time curve, Fig. 415 (*b*). The *area under this curve* between any two

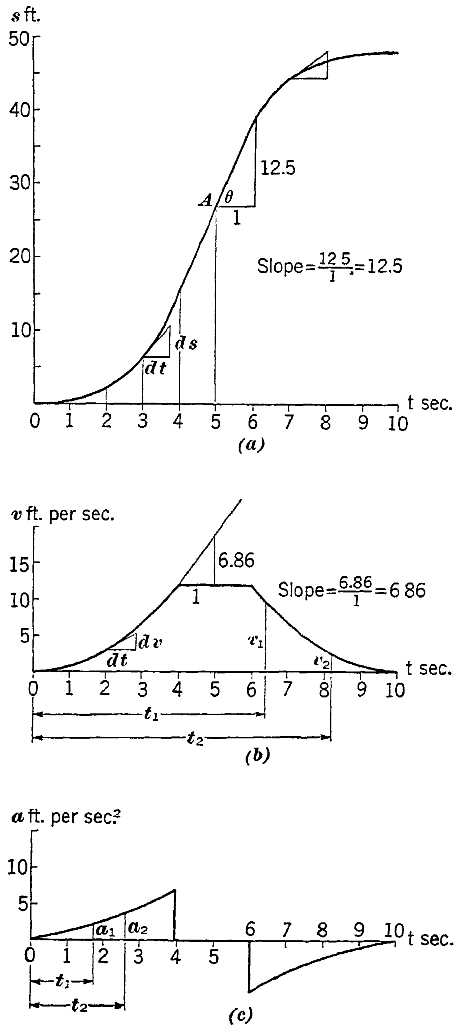


FIG. 415

speed ordinates v_1 and v_2, corresponding to the elapsed times t_1 and t_2, *is the distance traveled* by the particle during the time $(t_2 - t_1)$.

From Art. 115, $v=\frac{ds}{dt}$ or $ds=v\,dt$. The area under the curve is

$$\int_{s_1}^{s_2} ds=\int_{t_1}^{t_2} v\,dt$$

$$s_2-s_1=\int_{t_1}^{t_2} v\,dt$$

This equation indicates that the area inclosed by the curve and the two ordinates v_1 and v_2 is equal to the difference between the corresponding displacements, or the distance which the particle moves during the time (t_2-t_1).

The *slope of the tangent to the speed-time curve* at any point, or $\frac{dv}{dt}$, is the *instantaneous value of the acceleration* of the particle for the particular instant.

Acceleration-Time Curve.—If the values of the slope (magnitude of the acceleration) of the speed-time curve are plotted as ordinates against the corresponding values of the elapsed time as abscissae, the points will locate the acceleration-time curve, Fig. 415 (*c*). The area under this curve between any two ordinates a_1 and a_2 is given by the following equation:

$$\text{Area}=\int_{t_1}^{t_2} a\,dt$$

But, from Art. 116, $a=\frac{dv}{dt}$ or $a\,dt=dv$. Hence,

$$\int_{v_1}^{v_2} dv=\int_{t_1}^{t_2} a\,dt$$

or

$$v_2-v_1=\int_{t_1}^{t_2} a\,dt$$

This equation indicates that the area inclosed by the curve and the ordinates a_1 and a_2 is equal to the change of speed which takes place during the time (t_2-t_1) while the particle moves the distance (s_2-s_1).

The foregoing discussions may be summarized in the following statements:

1. The slope of the displacement-time curve at any point is the instantaneous value of the speed.

2. The slope of the speed-time curve at any point is the magnitude of the instantaneous acceleration. The area under the speed-time curve between any two ordinates v_1 and v_2 is the distance moved by the particle during the time (t_2-t_1).

3. The area under the acceleration-time curve between any two ordinates a_1 and a_2 is the change of speed (v_2-v_1) of the particle during the time (t_2-t_1).

These conclusions, as derived, apply only to a particle which is moving along a straight-line path. If a particle is moving along any curved path, similar curves can be plotted by substituting the distance traveled along the curve for the linear displacement and substituting the tangential speed and acceleration for the linear speed and acceleration.

EXAMPLE 1

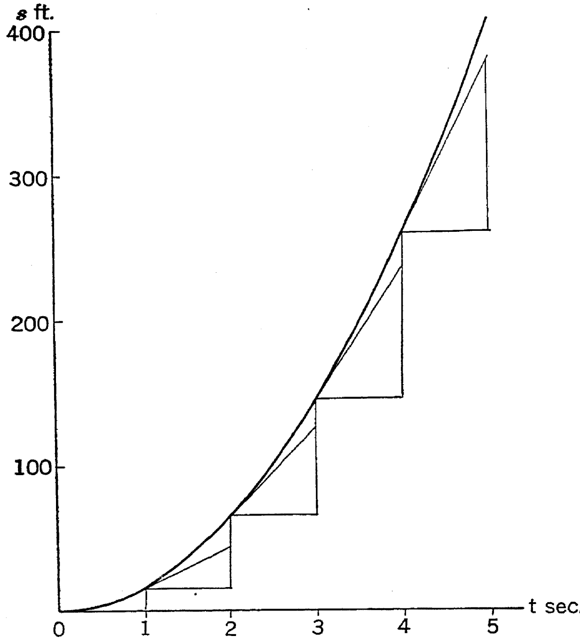
A certain particle moves along a straight-line path in such a manner that its displacements from a given point on the path after 1, 2, 3, 4, and 5 sec are 16.1, 64.4, 144.9, 257.6, and 402.5 ft. Construct the displacement-time, speed-time, and acceleration-time curves. Discuss the motion of the particle.

Fig. 416 (*a*) is the displacement-time curve. The slope of the curve can be found at any point by the construction indicated. Determine the slope at the end of each second. With these slopes as ordinates, construct the speed-time curve in Fig. 416 (*b*). This curve proves to be a straight line. Its slope is therefore constant. The value of this slope is determined as indicated by the construction in Fig. 416 (*b*). With this constant slope as an ordinate, the acceleration-time curve in Fig. 416 (*c*) is drawn. The motion is thus shown to be uniformly accelerated with a constant acceleration of 32.2 ft per sec per sec.

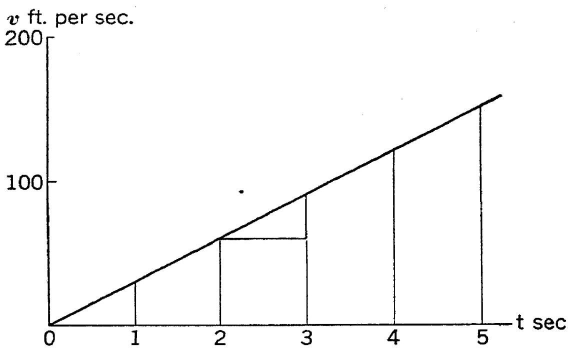
PROBLEMS

519. A body which has a speed of 44 ft per sec is brought to rest in 2 min. How far did it move during the 2 min? *Ans. 2,640 ft.*

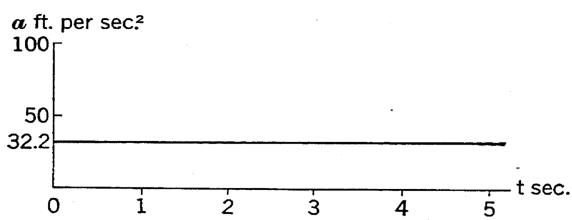
520. A train has a maximum speed of 60 mi per hr. If the train can be accelerated at the rate of 0.733 ft per sec per sec and decelerated by the brakes 0.88 ft per sec per sec, how long will it take to run 4 mi?


s ft.
400
300
200
100
0
1
2
3
4
5
t sec.

(a)


v ft. per sec.
200
100
0
1
2
3
4
5
t sec.

(b)


a ft. per sec.²
100
50
32.2
0
1
2
3
4
5
t sec.

(c)

FIG. 416

521. A bus is required to make a trip of 40 blocks, each 450 ft long. If the bus is permitted to stop only at every other block rather than at every block, how much time will be saved? The bus has a maximum speed of 15 mi per hr; and can be accelerated at the rate of 2 ft per sec per sec and decelerated by the brakes at the rate of 3 ft per sec per sec. Each stop is 15 sec long.

522. Fig. 417 is a diagrammatic sketch of a quick return mechanism such as is used to operate a shaper head or planer table. The large gear has a speed of 30 rpm. Construct an acceleration-time curve for the shaper table.

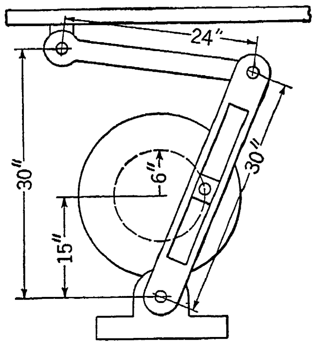


FIG. 417

REVIEW PROBLEMS

523. A particle moving along a circular path of 100-ft radius receives a linear displacement of 100 ft. What is the angle in radians between the radius vectors drawn to the two positions of the particle. *Ans.* $\frac{\pi}{3}$ rad.

524. A car travels north for 10 min at 30 mi per hr, east for 6 min at 20 mi per hr, and then south for 2 min at 45 mi per hr. Determine its displacement and its average velocity.

525. If the car in Problem 524 starts from rest and travels the same path in 15 min, what maximum velocity must it attain? What constant acceleration does it receive?

526. A particle is given an initial velocity of 40 ft per sec up a smooth 15° plane. What are the displacement and velocity of the particle 10 sec after starting up the plane?

527. An elevator is ascending with a velocity of 6 ft per sec and being accelerated upward at the rate of 1 ft per sec per sec at the instant a ball is dropped from a point 100 ft above the elevator. When and where will the ball and the elevator meet? *Ans. 2.28 sec; 83.7 ft.*

528. A ball is dropped from the top of a 500-ft building; and 1 sec later another ball is thrown upward from the ground. If the two balls pass at a point 150 ft from the ground, what was the initial velocity of the second ball?

529. A particle starts from rest and moves in a straight line with an acceleration $a = 2t^2$. How far will it travel in 3 sec?

530. Two boats leave port at the same time. Boat A travels south 30° east at 10 mi per hr. Boat B travels north 75° east at 15 mi per hr. How long after leaving port will the boats be 50 mi apart? What is the speed of boat A relative to boat B?

531. An airplane which can travel 180 mi per hr in still air is headed due east at a time when the wind is blowing 30 mi per hr from the southeast. Where will the plane be after 45 min have elapsed?

532. Two trains approach each other on parallel tracks. Train A is 2,000 ft long and is traveling at 45 mi per hr. Train B is 1,200 ft long and is traveling at 30 mi per hr. From the instant when the trains are a distance X apart until the rear ends of the trains are directly opposite each other 7.5 min elapse. Determine the distance X.

533. A ship traveling due west at 12 knots collides with a second ship traveling 24 knots north 30° east. With what velocity did the first ship strike the second? 1 knot = 1.152 mi per hr.

534. Train A travels due west at 45 mi per hr. Train B travels northwest at 60 mi per hr on a track which intersects the track of train A at a point C. If train A passes point C 15 min before train B, what is the displacement of train A relative to train B 20 min after train A passes point C? What is the velocity of train A relative to train B?

535. An airplane is to travel from field A to field B, which is 150 mi north 30° east from field A. The plane has an air speed of 250 mi per hr and the wind is blowing 30 mi per hr from north 15° west. Determine the time of flight and the direction in which the plane should be headed.

536. Airplane A, flying with an air speed of 280 mi per hr, starts due north from point B at the same instant at which airplane C, with an air speed of 400 mi per hr, starts from point D 150 mi southwest of point B. Determine the direction in which plane C should fly and the time required to intercept plane A.

537. A balloon which has attained an altitude of 1,000 ft is ascending at the rate of 800 ft per min and being carried due east by a wind of 40 mi per hr. If a small object is dropped from the balloon at the 1,000-ft elevation, when and where will this object strike the ground?

538. If a ship's gun is elevated 30° above the horizontal and the gun has a muzzle velocity of 1,200 ft per sec, what is its range? What is the maximum height attained by the shell?

539. If the ship in Problem 538 is to shell a point on a hill which is 1,000 ft above sea level, what is the greatest distance from the hill at which the ship could be stationed when the guns are pointed 30° above the horizontal? *Ans. 36,865 ft.*

540. A train starts from rest and moves with uniformly accelerated motion along a curve with a 3,000-ft radius. After 2.5 min it has attained a speed of 30 mi per hr. How far along the curve has it traveled? What are the normal and tangential components of its acceleration?

CHAPTER 14

KINEMATICS OF A RIGID BODY

126. General Statement.—A rigid body is any group of material particles arranged in a definite form and of such size that the dimensions of the body assume finite values.

When the motion of a rigid body is examined, it is readily seen that all its particles may or may not move in the same manner.

127. Types of Motion of Rigid Bodies.—The motion of a rigid body may be of any of the following kinds:

1. Translation
   (*a*) Rectilinear Translation
   (*b*) Curvilinear Translation
2. Rotation
3. Plane Motion

Translation.—When a rigid body moves in such a manner that any straight line drawn on the body in the plane of the motion remains parallel to its original position throughout the entire motion of the body, the motion is a translation.

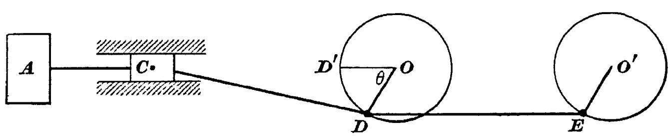


Fig. 418

Rectilinear Translation.—When a rigid body translates in such a manner that the particles of the body move along parallel straight lines, the motion of the body is rectilinear translation. Fig. 418 is a diagrammatic sketch of a locomotive piston, connecting-rod, and side rod. The motion of the piston A is a rectilinear translation. Each particle of the piston moves on a straight-line path, which is parallel to the path of each of the other particles.

Since the motion of every particle is exactly the same, the motion of any one of the particles determines the motion of the entire body. The equations developed in Chapter 13 for rectilinear

motion of a particle apply directly to the rectilinear translation of a rigid body.

Curvilinear Translation.—When a rigid body translates in such a manner that the particles of the body move along parallel curved lines, the motion is curvilinear translation. The motion of the side rod DE, Fig. 418, is curvilinear translation.

Rotation.—If a rigid body moves in such a manner that each particle of the body moves in a circular path around a fixed point, the body is rotating about that point. The crank DO, Fig. 418, is rotating about O. As the pin D moves to the position D', the crank DO moves through the angle θ. Rotation involves an angular change in position. The rotation may be about an axis which passes through the body or about an axis at some distance from the body.

Plane Motion.—When a body moves in such a manner that its motion is a combination of translation and rotation, its movement is called plane motion. The motion of the connecting-rod CD, Fig. 418, is plane motion. The pin D rotates around the axis O, while the pin C moves back and forth along a straight-line path. Since the motion of the rod is limited to one plane, the motion is designated as plane motion.

128. **Angular Displacement and Relation Between Linear and Angular Displacements.**—Consider any rigid body A, Fig. 419, which is rotating about an axis through O, perpendicular to the plane of the paper. Let P_1 and P_2 be any two particles of the body at distances ρ_1 and ρ_2 from O.

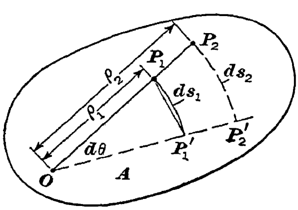


FIG. 419

The angle $d\theta$ through which the line OP_1P_2 swings as the body rotates about an axis through O is the angular displacement of the body A. This displacement may be measured in revolutions, degrees, or radians. *Every point in the body receives the same angular displacement.* When the displacement is measured in radians, it is given by the ratio

$$\frac{\text{length of arc}}{\text{length of radius}}=\frac{ds}{\rho_2}=\theta \text{ radians}$$

As particle P_1 moves to the position P_1', its linear displacement is given by the chord P_1P_1'. When the angle $d\theta$ is small, the chord

P_1P_1' can be taken as equal to the arc ds_1 without serious error. The lengths ds_1 and ds_2 can be expressed in terms of the angular displacement as follows:

$$ds_1 = \rho_1 \, d\theta \text{ and } ds_2 = \rho_2 \, d\theta$$

PROBLEMS

541. A particle on the rim of a pulley 10 ft in diameter moves 10 ft along the circumference of the pulley as the pulley turns on its shaft. What is the angular displacement of the pulley in radians and degrees? *Ans. 2 rad.; 114.6°.*

542. A wheel 12 ft in diameter turns through an angle of 75°. Compute the angular displacement of the wheel in radians. What is the displacement of a particle on the rim of the wheel along the path traced by the particle as the wheel turns through the 75° angle? What is the linear displacement of the same particle?

129. **Angular Speed and Velocity.**—*Angular speed is the time rate of angular motion without regard for direction. Angular velocity is the time rate of change of angular displacement.* The unit of angular velocity is any convenient unit of angular displacement per unit of time, as radians per second, revolutions per minute, or degrees per second.

If $d\theta$ is the angular displacement in time dt by the radius vector ρ_1, Fig. 419, then $\omega = \frac{d\theta}{dt}$ is the instantaneous angular velocity, in radians per second, of the entire body A about an axis through O. If the body turns through equal angles in equal periods of time, the uniform angular velocity is given by $\omega = \frac{\theta}{t}$, where θ is the angle, expressed in radians, through which the body turns in the time t. If the motion is not uniform, then $\omega = \frac{\theta}{t}$ expresses the average velocity in radians per second. Angular velocity is represented vectorially in the following manner. A vector, which represents the magnitude of the velocity in any convenient units, is drawn parallel to the axis of rotation and pointing in the direction toward which a right-hand screw would move if it were turned in the direction of the rotation.

Since angular velocity is a function of angular displacement, it follows that *all points on a rigid body have the same angular velocity.*

PROBLEMS

543. A pulley A, 36 in. in diameter, is turning at the rate of 90 rpm. What is the total angular displacement in 3 min? What is the angular velocity in radians per sec? *Ans.* *1,695 rad.; 3π rad.*

544. If the pulley of Problem 543 starts from rest and makes 75 complete revolutions in 40 sec, what is its average angular velocity in radians per sec? What angular velocity will it attain at the end of the 40 sec?

130. Angular Acceleration.—*Angular acceleration is the time rate of change of angular velocity.* The instantaneous angular acceleration is given by $\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$, where $d\omega$ is the differential change in the angular velocity which occurs during the time dt and $d\theta$ is the angular displacement during the same time.

If the angular acceleration remains constant during a period of time t, while the velocity is changing from ω_0 to ω, the angular acceleration is given by $\alpha=\frac{\omega-\omega_0}{t}$.

If the acceleration is variable, then the last equation gives the average value of the angular acceleration.

The unit of angular acceleration depends on the units which are used to express angular velocity. Angular acceleration may be expressed as radians per second per second, revolutions per second per second, or degrees per second per second.

Angular acceleration is a function of angular velocity and also of angular displacement. Therefore, *every point on a rigid body has the same angular acceleration.*

If dt is eliminated from the equations $\omega=\frac{d\theta}{dt}$ and $\alpha=\frac{d\omega}{dt}$, we obtain the equation $\omega\, d\omega=\alpha\, d\theta$.

The basic differential equations which may be used to solve any rotational kinematic problem, if the relationships of the variables are known, are

$$\omega=\frac{d\theta}{dt}$$

$$\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$$

$$\omega\, d\omega=\alpha\, d\theta$$

PROBLEMS

545. A flywheel which was turning at the rate of 200 rpm was brought to rest in 2 min. What was its average angular velocity, while coming to

rest? What angular acceleration did it receive? *Ans. 10.46 rad. per sec; 0.174 rad. per sec².*

546. A large pulley makes 5 revolutions in 4 min. What is its angular displacement in radians? What is the angular velocity in radians per sec?

547. If the pulley in Problem 546 started from rest and received a constant acceleration in turning the 5 revolutions during the 4-min period, what maximum angular velocity did it develop? What was its angular acceleration?

131. Relationship Between Linear and Angular Velocities and Tangential and Angular Accelerations.—In Fig. 420, A is any body which is rotating about an axis through any point O and perpendicular to the plane of the paper. The particle P has a linear velocity v in a tangential direction. The relationship between the motion of this particle along its curved path and the angular motion of the rigid body about the center of rotation O can be shown in the following manner.

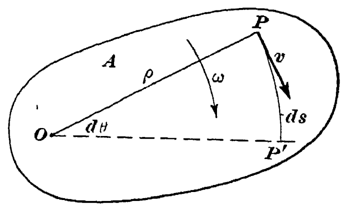


FIG. 420

Let ds represent the distance traveled by the particle P along the arc of radius ρ in going from P to P'.

$$ds = \rho\, d\theta \tag{1}$$

$$\frac{ds}{dt} = \rho \frac{d\theta}{dt}$$

From Art. 115, $\frac{ds}{dt} = v$; and, from Art. 129, $\frac{d\theta}{dt} = \omega$. Thus,

$$v_t = \rho\, \omega \tag{2}$$

where v_t is the tangential velocity of any particle of the rigid body any distance ρ from the axis of rotation and ω is the instantaneous value of the angular velocity of the rotating body.

$$v_t = \rho\, \omega$$

$$\frac{dv_t}{dt} = \rho \frac{d\omega}{dt}$$

From Art. 116, $\frac{dv_t}{dt} = a_t$; and, from Art. 130, $\frac{d\omega}{dt} = \alpha$. Hence,

$$a_t = \rho\, \alpha \tag{3}$$

where a_t is the tangential acceleration of a particle a distance ρ from the axis of rotation and α is the angular acceleration of the body about the same axis.

It will be observed from equations (1), (2), and (3) that *each of the linear properties of motion is* ρ *times the corresponding angular function.*

These equations may also be applied to any given particle of a body which moves along any plane curve. When so used, ρ is the instantaneous value of the radius vector from the particle to the instantaneous center of curvature of the curve; v is the instantaneous tangential speed of the particle; ω is the instantaneous angular velocity; and α is the instantaneous angular acceleration of the radius vector drawn from the particle to the center of curvature.

PROBLEMS

548. A body is traveling around a circular path in such a manner that a certain particle of the body moves in a circle of 5-ft radius. The tangential velocity of this particle is 50 ft per sec. What is the angular velocity of the body in radians? Express the angular velocity in rpm. *Ans. 10 rad. per sec; 95.4 rpm.*

549. If the body of Problem 548 starts from rest and after 5 min it has attained a speed of 50 rpm, what is the maximum tangential velocity of the particle? What tangential acceleration did the particle receive? What angular acceleration did the body receive?

550. If 100 ft of the circumference of a friction wheel 5 ft in diameter comes in contact with a smaller wheel during the first 40 sec after the wheels start from rest, what is the tangential velocity of a particle on the rim of the larger wheel, and what is the angular velocity of the wheel at the end of the 40-sec period? What tangential acceleration has a particle on the rim received? What angular acceleration has the wheel been given?

551. A ball 6 in. in diameter starts from rest and rolls down an inclined plane 20 ft long in 10 sec. Compute: (*a*) the average rectilinear velocity of the ball; (*b*) the average angular velocity; (*c*) the rectilinear velocity of the ball at the end of the incline; (*d*) the angular velocity of the ball when it is two-thirds of the way down the incline.

552. Two cable drums are keyed to the same shaft. Drum A is 6 ft in diameter and supports a weight B, which hangs from a cable wound on the drum. Drum C, 4 ft in diameter, supports a weight D in the same manner, but on the opposite side of the shaft from weight B. If the system starts from rest and the weight B attains a velocity of 30 ft per sec in 5 sec, what are the velocity and acceleration of weight D? What is the normal acceleration of a particle on the rim of the smaller pulley?

553. A rod 5 ft long turns in a horizontal plane about a vertical axis through one end of the rod. If the angular velocity of the rod increases 2 rad. every 5 sec, what are the normal and tangential accelerations of a point on the rod 3 ft from the axis, after the rod has been in motion 4 sec? *Ans. 7.67 ft per sec*^{2}*; 1.2 ft per sec*^{2}*.*

132. Constant Angular Acceleration.—When the law which governs the angular acceleration of a body is known, the special equations which express the relationships existing between θ, ω, α, and t may be derived from the general differential equations for kinematic rotation. Thus,

$$\omega=\frac{d\theta}{dt},\ \alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2},\ \text{and}\ \omega\,d\omega=\alpha\,d\theta$$

The most frequently encountered law is that expressed by $\alpha=c$, a constant; this relationship indicates a constant angular acceleration throughout the motion. Formulas *for constant angular acceleration, $\alpha=c$,* will now be derived by integration of the basic differential equations.

$$\int d\omega=\int\alpha\,dt$$

$$\omega=\alpha\,t+C_1$$

When $t=0$, $\omega=\omega_0$. Therefore $C_1=\omega_0$ and

$$\omega=\omega_0+\alpha\,t \tag{1}$$

Since $\omega=\dfrac{d\theta}{dt}$,

$$\int d\theta=\int\omega_0\,dt+\int\alpha\,t\,dt$$

$$\theta=\omega_0\,t+\frac{\alpha\,t^2}{2}+C_2$$

When $t=0$, $\theta=0$. Hence, $C_2=0$ and

$$\theta=\omega_0\,t+\frac{1}{2}\alpha\,t^2 \tag{2}$$

$$\int\omega\,d\omega=\int\alpha\,d\theta$$

$$\frac{\omega^2}{2}=\alpha\,\theta+C_3$$

When $\theta=0$, $\omega=\omega_0$. Therefore, $C_3=\dfrac{\omega_0^2}{2}$ and

$$\omega^2=\omega_0^2+2\,\alpha\,\theta \tag{3}$$

By eliminating α from equations (1) and (2), we obtain

$$\theta=\left(\frac{\omega+\omega_0}{2}\right)t \tag{4}$$

These equations may also be obtained directly from the equations of Art. 117 by substitution of $s=\rho\,\theta$, $v=\rho\,\omega$, and $a=\rho\,\alpha$.

Equations (1), (2), (3), and (4) are true only when $\alpha=c$. As in Art. 117 consistency in the use of signs is necessary. Usually

the *direction of the initial angular motion is taken as the positive direction.* If the quantities θ, ω, and α are in the same direction as the initial motion, they are given the positive sign. If in the opposite direction, they are given the negative sign.

EXAMPLE

A flywheel turning 120 rpm has its speed reduced to 30 rpm in 45 sec. What is its angular acceleration, and how many revolutions does the wheel make during the 45 sec?

$$\omega_0=\frac{120\times 2\pi}{60}=4\pi \text{ and } \omega=\frac{30\times 2\pi}{60}=\pi$$

$$\omega=\omega_0+\alpha\, t$$

$$\pi=4\pi+(-\alpha)45$$

$$\alpha=0.209 \text{ rad. per sec}^2$$

$$\theta=\left(\frac{\pi+4\pi}{2}\right)45$$

$$\theta=353 \text{ rad. or } 56.2 \text{ rev.}$$

Another solution follows:

$$\int_{4\pi}^{\pi} d\omega=\int_0^{45}(-\alpha)\, dt$$

$$\alpha=0.209 \text{ rad. per sec}^2$$

$$\int d\omega=\int \alpha\, dt$$

$$\omega=\alpha\, t+C_1$$

When $t=0$, $C_1=\omega_0$. Hence,

$$\omega=\alpha\, t+\omega_0=\omega_0+\alpha\, t$$

Since $\omega=\dfrac{d\theta}{dt}$,

$$\int_0^{\theta} d\theta=\int_0^{45}(-0.209)\, t\, dt+\int_0^{45}4\pi\, dt$$

$$\theta=353 \text{ rad.}$$

PROBLEMS

554. A flywheel rotating with a constant angular acceleration attains a speed of 300 rpm in 30 sec. What is its angular acceleration? How many revolutions does the wheel make in a half-minute? *Ans.* $\frac{\pi}{3}$ *rad. per sec*^{2}*; 75 rev.*

555. A wheel has a speed of 800 rpm when a brake is applied, which reduces the speed at the rate of 4 rad. per sec per sec. How long will the

wheel continue to turn, and how many revolutions will it make in coming to rest?

556. A flywheel has its speed increased from 50 rpm to 180 rpm in 80 sec. The diameter of the wheel is 6 ft. What are the angular acceleration and the tangential acceleration of a point on the rim of the wheel? What is the maximum tangential velocity of the point on the rim?

557. A pulley 4 ft in diameter is driven from a pulley 18 in. in diameter. The smaller pulley is running at 150 rpm. What is the angular velocity of the larger pulley? What is the tangential velocity of a point on the rim of the larger pulley?

558. A flywheel which has a speed of 120 rpm receives an acceleration of 30 rpm per min. What rpm will it attain in 40 sec? *Ans. 140 rpm.*

133. Variable Angular Acceleration.—When the angular acceleration is not constant, the basic differential equations of Art. 130 must be employed.

EXAMPLE 1

A rotating object has an angular acceleration $\alpha=-\theta$. It has an initial angular velocity of 60 rad. per sec. (*a*) What is its angular velocity after 8 revolutions? (*b*) What is the elapsed time for the 8 revolutions?

$$\int_{60}^{\omega} \omega\, d\omega=\int_{0}^{16\pi} \alpha\, d\theta=\int_{0}^{16\pi} -\theta\, d\theta$$

$$\omega=32.8 \text{ rad. per sec}$$

$$\int \omega\, d\omega=\int \alpha\, d\theta=\int -\theta\, d\theta$$

$$\frac{\omega^2}{2}=-\frac{\theta^2}{2}+C_1$$

When $\theta=0$, $\omega=\omega_0$. Therefore, $C_1=\frac{\omega_0^2}{2}=\frac{3{,}600}{2}$ and

$$\omega^2=3{,}600-\theta^2$$

$$\omega=\frac{d\theta}{dt}$$

$$\int_0^t dt=\int_0^{16\pi} \frac{d\theta}{\omega}=\int_0^{16\pi} \frac{d\theta}{\sqrt{3{,}600-\theta^2}}$$

$$t=\sin^{-1}\frac{\theta}{60}\Big]_0^{16\pi}=\sin^{-1}\frac{16\pi}{60}=\sin^{-1} 0.8373$$

$$t=0.992 \text{ sec}$$

PROBLEMS

559. A heavy sphere is attached to a steel wire and suspended from a ceiling. The angular acceleration of the sphere is $\alpha = -10\,\theta$. The sphere is turned through 720° from its static position and released. (*a*) With what angular velocity will the sphere pass through its original static position? (*b*) How long after release will it pass its original static position?

560. A wheel turns with an angular acceleration of $\alpha = t - 4$ rad. per sec per sec. It had an initial angular velocity of 5 rad. per sec in the direction of the initial acceleration. (*a*) What is the angular velocity of the wheel? (*b*) What is its angular displacement from the initial position after 10 sec and after 15 sec?

134. **Plane Motion.**—When a body moves so that each point in the body remains at a constant distance from some fixed reference plane, the body is then executing plane motion.

The motion may consist of a rotation around an axis perpendicular to the reference plane; a translation parallel to the reference plane; or a motion which is a combination of this rotation and translation.

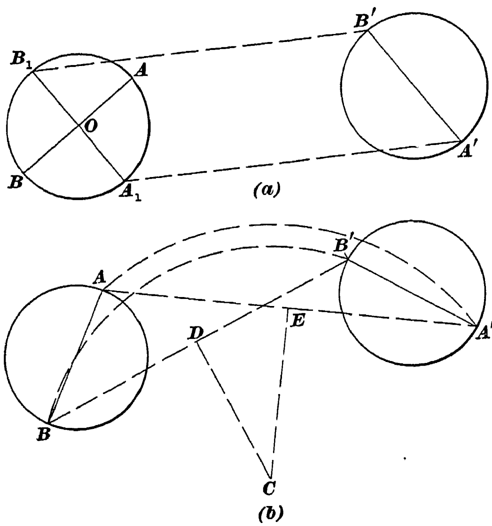


FIG. 421

The motion of the connecting-rod on a stationary steam engine and the motion of the landing wheels on an airplane at the instant the airplane takes off are examples of plane motion.

In Fig. 421 (*a*), *A* and *B* represent two points on a diameter of an airplane landing wheel at the instant the craft leaves the ground;

and A' and B' are the same points an instant later. This displacement may be accomplished by a rotation about O to the position A_1B_1, plus a rectilinear translation, as indicated by the dotted lines.

In Fig. 421 (*b*) the same displacement is obtained by a single rotation about the axis through the point C, as indicated by the construction lines.

If in Fig. 421 (*b*) the displacement were such as to cause the perpendiculars CD and CE to be parallel at all times and not intersect, the motion would be a rectilinear translation.

135. Instantaneous Center.—In Art. 134 it was shown that any plane motion was equivalent to a rotation about some point or a rotation plus a translation.

It is sometimes desirable to determine the point about which a body is rotating at any given instant during plane motion. If the directions of the velocities of any two points on the body are known for any particular instant, the instantaneous center, or point about which rotation is taking place, can be determined.

Let v_A and v_B, Fig. 422, be the directions of the instantaneous velocities of the points A and B on a rigid body moving with plane motion. If at this instant the body is rotating about some point O as a center, the points A and B must move in circular paths for which O is the center and the velocities v_A and v_B must be tangent to the circles. Therefore, if lines are drawn through A and B normal to the tangential velocities v_A and v_B, the center of rotation will lie on each of the normals or at their intersection O, which is the instantaneous center.

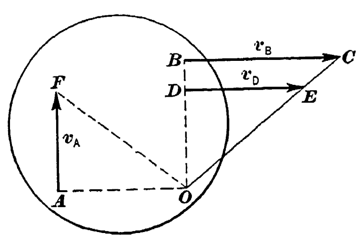


FIG. 422

If the motion happens to be rectilinear translation at any instant, the instantaneous center at that instant is at infinity.

For the general case of plane motion the instantaneous center is not fixed but changes from one instant to the next. If the instantaneous center is a point on the body, that point of the body must have zero velocity. Although the instantaneous center must be a point at which the instantaneous velocity is zero, the instanta-

neous acceleration of the center is not necessarily zero. This fact can be demonstrated in the following manner.

If the body is rotating about O as a center and v_B is the vector representing the velocity of point B, then v_D must be the vector which represents the velocity of point D. The center O is also on the normal OB; and, if it has any velocity, its velocity vector must also be parallel to v_B. Likewise, O is on the normal OA; and, if O has any velocity, the velocity vector of O would have to be parallel to the vector v_A. Since the point O cannot move parallel to v_A and parallel to v_B at the same instant, the point O must remain stationary.

Since all parts of the body have the same instantaneous angular velocity ω, it follows from Art. 131 that the tangential velocities of all points on the body are directly proportional to their radial distances from the instantaneous center, or $v=\rho\,\omega$. For the instantaneous center, $\rho=0$; therefore, its tangential velocity is zero.

PROBLEMS

561. Locate the instantaneous center for the connecting-rod shown in Fig. 423. *Ans. 3.37 ft above B.*

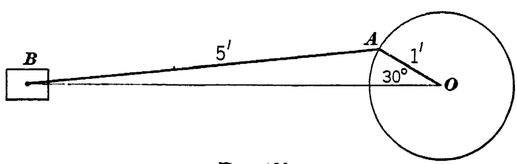


FIG. 423

562. Where will the instantaneous center of the rod of Problem 561 be after the crank has turned through 90° in a clockwise direction?

563. Locate the instantaneous center for the link AB, Fig. 429, after the link CB has turned 30° in a counter-clockwise direction.

564. Determine the absolute velocity of the sliding block A in Problem 563.

136. Velocity During Plane Motion.—If the absolute velocity of any given point on a body which is executing plane motion is known, the absolute velocity of any other point on the body can be obtained by means of the relative motion theorem of Art. 119.

Fig. 424 represents a body which is executing plane motion in the plane of the paper. A is any given point on the body which has a known instantaneous absolute velocity v_A. B is any other

point on the body a distance r from A. Since B cannot approach A, the velocity of B relative to A must be in the tangential direction or normal to the line connecting A and B. If B has an angular velocity about A of ω, the velocity of B relative to A is $r\,\omega$.

By the theorem of Art. 119,

$$v_B = v_A \nrightarrow v_{\frac{B}{A}}$$

$$v_B = v_A \nrightarrow r\,\omega \text{ (vector sum)}$$

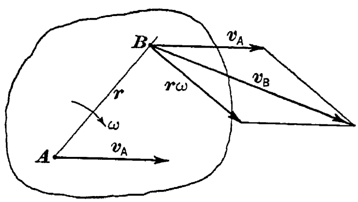


Fig. 424

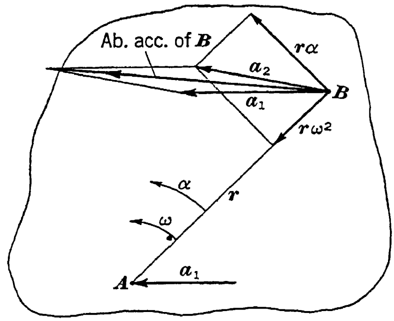


Fig. 425

Algebraically, the absolute velocity of a point, such as B, can generally be obtained most easily by summing the components of the absolute and relative velocities parallel to the X and Y axes and then obtaining the resultant velocity from

$$v_B = \sqrt{(\Sigma v_x)^2 + (\Sigma v_y)^2}$$

137. Acceleration During Plane Motion.—The absolute acceleration of any point on a body executing plane motion can also be obtained by means of the relative motion theorem of Art. 119.

Fig. 425 represents a body executing plane motion in the plane of the paper. Point A has an instantaneous absolute acceleration a_1, and B is a point at distance r from A. The instantaneous values of the angular velocity and the angular acceleration are ω and α. The acceleration of B relative to A can be found by getting the resultant of the tangential and normal components of its acceleration relative to A. These components are, respectively, $r\,\alpha$ and $r\,\omega^2$. The acceleration of B relative to A is then a_2, which is given by the parallelogram construction shown in Fig. 425.

By the theorem of Art. 119,

$$a_B = a_A \nrightarrow a_{\frac{B}{A}}$$

$$a_B = a_1 \nrightarrow a_2 \text{ (vector sum)}$$

Algebraically this result can be obtained by summing the acceleration components parallel to the X and Y axes and then getting the resultant acceleration from the following equation:

$$a_B = \sqrt{(\Sigma a_x)^2 + (\Sigma a_y)^2}$$

EXAMPLE

The center of the wheel shown in Fig. 426 (a) has a velocity of 5 ft per sec and an acceleration of 3 ft per sec per sec parallel to the horizontal plane. Determine the absolute velocity and acceleration of a point A on the rim and the acceleration of the instantaneous center B.

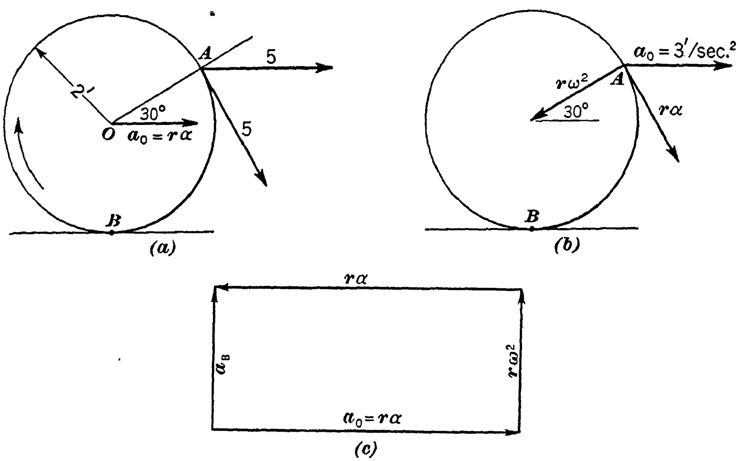


FIG. 426

Since the center of the wheel is moving parallel to the horizontal plane with a velocity of 5 ft per sec, every other point on the wheel, such as A, moves parallel to the plane with a velocity of 5 ft per sec. Point A also moves in a tangential direction with a velocity of 5 ft per sec.

$$\Sigma v_x = 5 + 5 \times 0.5 = 7.5$$
$$\Sigma v_y = 5 \times 0.866 = 4.33$$

$$v_A = \sqrt{7.5^2 + 4.33^2} = 8.66 \text{ ft per sec}$$

In Fig. 426 (b), the point A is shown with its acceleration components. Every point on the wheel receives an acceleration

equal to the acceleration of the center. The point A also receives an acceleration in the tangential and normal directions.

$$v = r\,\omega \text{ and } \omega = \frac{5}{2} = 2.5 \text{ rad. per sec}$$

$$a_n = r\,\omega^2 = 2 \times 2.5^2 = 12.5 \text{ ft per sec per sec}$$

$$a_t = 3 = r\,\alpha$$

$$\Sigma a_x = 3 + 3 \times 0.5 - 12.5 \times 0.866 = -6.33$$

$$\Sigma a_y = -12.5 \times 0.5 - 3 \times 0.866 = -8.85$$

$$a_A = \sqrt{(-6.33)^2 + (-8.85)^2} = 10.87 \text{ ft per sec per sec}$$

The instantaneous center is always a point with zero velocity but usually is not a point which remains fixed in space; therefore, it may have an acceleration.

In Fig. 426 (a) the point B is the instantaneous center, and the center of the wheel has an acceleration $a = 3$ ft per sec per sec. The acceleration of B is found from the following equation and the acceleration diagram in Fig. 426 (c).

$$a_B = a_O \nrightarrow a_{\frac{B}{O}}$$

$$a_B \uparrow = \underset{\longrightarrow}{r\alpha} \nrightarrow r\,\omega^2 \uparrow \nrightarrow \underset{\longleftarrow}{r\alpha}$$

The equation and the acceleration diagram show that the acceleration a_B is simply the normal acceleration of B relative to O, or $r\,\omega^2$.

PROBLEMS

565. If the wheel in Fig. 426 (a) is moving with a uniform angular velocity of 2 rad. per sec, clockwise, what are the velocity and acceleration of point B? *Ans. 0; 8 ft per sec².*

566. With the wheel of Fig. 426 (a) moving as in Problem 565, what are the velocity and acceleration of point A relative to point B?

567. By means of the result of Problem 561, determine the velocity of the cross-head B, shown in Fig. 423. The crank OA is turning at 120 rpm.

138. Linkages.—The applications of the principles of plane motion in the solutions of certain velocity and acceleration problems which are encountered in machine design will now be illustrated.

EXAMPLE 1

The point A on rod AB, Fig. 427, has a velocity of 10 ft per sec and an acceleration of 5 ft per sec per sec to the right. Determine the velocity and acceleration of the point B when the rod is in the position shown.

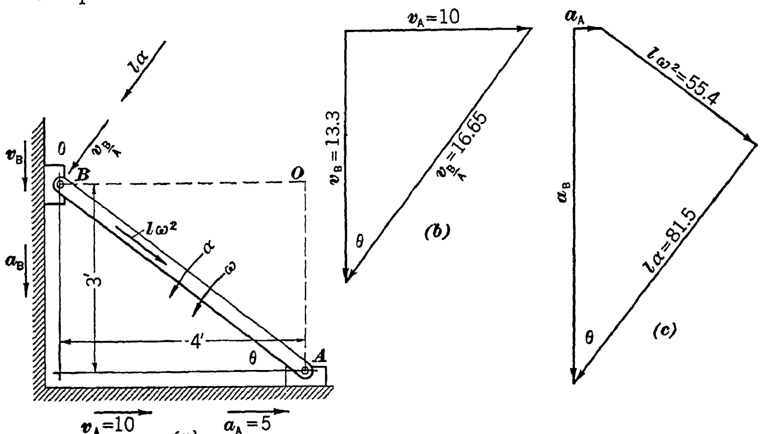


FIG. 427

Since A and B are two points on a rigid body, the only motion which B can have relative to A is to rotate about A. Its instantaneous velocity relative to A is $v_{\frac{B}{A}}$ normal to the rod AB. The absolute velocities and accelerations of points A and B must be parallel to the respective surfaces because A and B are constrained to remain in contact with the surfaces. The instantaneous center of the rod is at O (see Art. 135).

$$v_B = v_A \times \frac{4}{3} = \frac{10 \times 4}{3} = 13.3 \text{ ft per sec}$$

$$\omega = \frac{10}{3} = 3.33 \text{ rad. per sec}$$

The velocity v_B can also be obtained from the relative motion theorem, Art. 119, by a graphical solution or an analytical solution.

$$\downarrow v_B = \underset{\rightarrow}{v_A} \nrightarrow \swarrow v_{\frac{B}{A}} \text{ (vector sum)}$$

$$\downarrow v_B = \underset{\rightarrow}{v_A} \nrightarrow \swarrow l\,\omega$$

$$\downarrow v_B = \underset{\rightarrow}{10} \nrightarrow \swarrow 5 \times 3.33$$

For the graphical solution, construct the velocity triangle in Fig. 427 (*b*) to scale and obtain v_B.

For an analytical solution apply the sine law to Fig. 427 (*b*).

$$\frac{v_B}{\frac{4}{5}}=\frac{10}{\frac{3}{5}} \text{ and } v_B=13.3 \text{ ft per sec}$$

To obtain a_B, again apply the relative motion theorem, Art. 119.

$$a_B= a_A \mathrel{+\!\!\!\rightarrow} a_{\frac{B}{A}}$$

$$\downarrow a_B= \underset{\rightarrow}{5} \mathrel{+\!\!\!\rightarrow} \searrow l\,\omega^2 \mathrel{+\!\!\!\rightarrow} \swarrow l\,\alpha \text{ (vector sum)}$$

$$\downarrow a_B= \underset{\rightarrow}{5} \mathrel{+\!\!\!\rightarrow} \searrow 5\times 3.33^2 \mathrel{+\!\!\!\rightarrow} \swarrow 5\alpha$$

For a graphical solution, draw the acceleration polygon in Fig. 427 (*c*) to scale and obtain a_B.

For an analytical solution, examination of the acceleration polygon will indicate that if a horizontal summation of the acceleration vectors is made the following equation will be obtained:

$$\Sigma a_H=0$$

$$5+55.4\times\frac{4}{5}-5\alpha\times\frac{3}{5}=0$$

$$\alpha=16.44 \text{ rad. per sec}^2$$

In a similar manner a vertical summation will give:

$$a_B=l\,\omega^2\times\frac{3}{5}+l\,\alpha\times\frac{4}{5}$$

$$a_B=5\times 3.33^2\times\frac{3}{5}+5\times 16.44\times\frac{4}{5}$$

$$a_B=99.03 \text{ ft per sec}^2$$

EXAMPLE 2

In Fig. 428, *AB* and *BC* are two links oı equal length. Link *AB* has a fixed pin at *A*, about which it revolves in a counterclockwise direction with an angular velocity of 4 rad. per sec. What are the absolute velocity and acceleration of the sliding block *C*?

Since *A* and *B* are two points on a rigid body, the only motion which *B* can have relative to *A* is to rotate about the fixed point *A*.

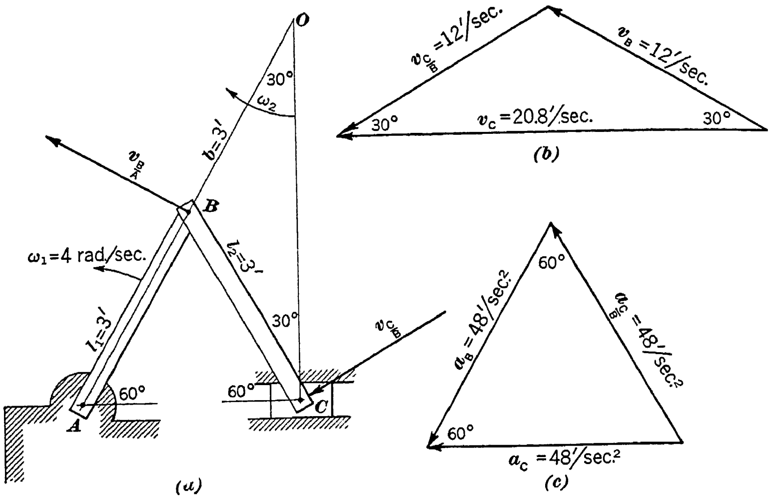


FIG. 428

Thus, $v_B = v_{\frac{B}{A}} = l_1\,\omega_1 = 3\times 4 = 12$ ft per sec normal to AB. Because of the construction of the linkage, the absolute velocity and acceleration of C must be along the line AC. Since the directions of the absolute velocities of points B and C are known, the instantaneous center for the link BC is easily located at O, Fig. 428 (a). From the geometry of the triangle OBC, $BO = b = 3$ ft and $OC = 5.2$ ft.

$$v_C = \frac{12\times 5.2}{3} = 20.8 \text{ ft per sec}$$

$$\omega_2 = \frac{v_C}{OC} = \frac{20.8}{5.2} = 4 \text{ rad. per sec, clockwise}$$

The velocity v_C can also be obtained from the relative motion theorem, Art. 119, by either a graphical solution or an analytical solution.

$$\underset{\leftarrow}{v_C} = \nwarrow v_B \nrightarrow \swarrow v_{\frac{C}{B}} \text{ (vector sum)}$$

$$\underset{\leftarrow}{v_C} = \nwarrow v_B \nrightarrow \swarrow l_2\,\omega_2$$

$$\underset{\leftarrow}{v_C} = \nwarrow 12 \nrightarrow \swarrow 3\times 4$$

For the graphical solution of this equation, construct the velocity triangle in Fig. 428 (b) to scale and obtain v_C.

For the analytical solution of the equation, apply the sine law or take a horizontal summation in Fig. 428 (*b*).

$$\frac{v_C}{0.866}=\frac{12}{0.5}$$
$$v_C=20.8 \text{ ft per sec}$$

From a horizontal summation,

$$v_C=12\times 0.866+12\times 0.866$$
$$v_C=20.8 \text{ ft per sec}$$

To obtain a_C, again apply the relative motion theorem, Art. 119. Since link AB has a constant angular velocity of 4 rad. per sec, the point B has zero tangential acceleration but a normal acceleration $l_1\,\omega_1^2$ along AB.

$$a_B=a_A \nrightarrow a_{\frac{B}{A}}$$
$$a_B=0 \nrightarrow l_1\,\omega_1^2$$
$$a_B=\swarrow 3\times 4^2=\swarrow 48 \text{ ft per sec}^2$$

Since B and C are points on the rigid body BC, the motion of C relative to B is a rotation about B. Therefore, the acceleration of C to B can be represented by its normal component $l_2\,\omega_2^2$ along BC and the tangential component $l_2\,\alpha$ perpendicular to BC.

$$a_C = a_B \nrightarrow a_{\frac{C}{B}}$$
$$\underset{\leftarrow}{a_C} = \swarrow 48 \nrightarrow \nwarrow l_2\,\omega_2^2 \nrightarrow / \, l_2\,\alpha \text{ (vector sum)}$$
$$\underset{\leftarrow}{a_C} = \swarrow 48 \nrightarrow \nwarrow 3\times 4^2 \nrightarrow / \, 3\,\alpha$$

Study of this equation will show that the quantity 3α must be zero if a closed vector diagram is to be drawn.

For a graphical solution of this equation, construct the acceleration triangle in Fig. 428 (*c*) to scale and obtain a_C.

For an analytical solution, apply the sine law to Fig. 428 (*c*) or take a horizontal summation.

$$\frac{a_C}{0.866}=\frac{48}{0.866}$$
$$a_C=48 \text{ ft per sec}^2$$

From a horizontal summation,

$$a_C=48\times 0.5+48\times 0.5=48 \text{ ft per sec}^2$$

PROBLEMS

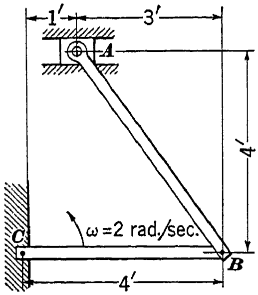


FIG. 429

568. By means of the relative motion theorem, determine the velocity of the cross-head B of Problem 567. *Ans. 7.38 ft per sec.*

569. If link AB of Fig. 428 has an angular velocity of 3 rad. per sec and an angular acceleration of 1 rad. per sec per sec, both in a counterclockwise direction, what are the absolute velocity and acceleration of the block C?

570. If the arm BC, Fig. 429, rotates about C with a constant angular velocity of 2 rad. per sec, what are the absolute velocity and acceleration of the sliding block A?

REVIEW PROBLEMS

571. A motor which is running at 600 rpm when the power is turned off is brought to rest in 50 sec. What is its angular displacement in radians while coming to rest? *Ans.* 500π *rad.*

572. Pulley A, which is 2 ft in diameter, drives pulley B, 36 in. in diameter. If pulley A is running at 120 rpm, what are the angular velocity and tangential velocity of a point 15 in. from the center of pulley B?

573. If the pulleys of Problem 572 start from rest and pulley A attains a speed of 180 rpm in 50 sec, what are the angular velocity and acceleration of pulley B? How many revolutions will pulley B make during the 50 sec?

574. What normal and tangential accelerations does a point on the rim of pulley B, Problem 573, have? *Ans. 236.6 ft per sec*^{2}*; 0.377 ft per sec*^{2}.

575. A body rotates with an angular acceleration $\alpha = 3t^2 + 5$. If it has an initial angular velocity of 3 rad. per sec, determine: (*a*) its angular displacement and (*b*) its angular velocity in rpm after 5 sec.

576. A wheel 5 ft in diameter rolls along a horizontal plane at the rate of 120 rpm. By the instantaneous center method, determine the absolute velocity of a point on the rim of the wheel. The point is 15° above the horizontal on the side toward which the wheel is rolling.

577. What is the absolute acceleration of the point mentioned in Problem 576, if the wheel is given an angular acceleration of 1 rad. per sec per sec? *Ans. 392 ft per sec*^{2}.

578. In Problem 576 what is the acceleration of the point on the rim of the wheel which is in contact with the plane?

579. Check the result of Problem 577 by using the instantaneous center of the wheel as the point of reference, A mentioned in Fig. 425, Art. 137.

580. If the angular velocity of arm BC, Fig. 430, is 3 rad. per sec and its angular acceleration is 2 rad. per sec per sec, what are the absolute velocity and acceleration of the sliding block A?

581. Determine the absolute velocity and acceleration of block B, Fig. 431, if block A has a velocity of 10 ft per sec to the right and an acceleration of 5 ft per sec per sec to the left.

582. Determine the absolute velocity and acceleration of the cross-head B for the position shown in Fig. 432, if the crank OA is rotating at a constant clockwise speed of 90 rpm.

583. The crank AB, Fig. 433, has a constant clockwise velocity of 2 rad. per sec. Determine the absolute velocity of pin C, the angular velocity of the crank CD, and the angular velocity of bar BC.

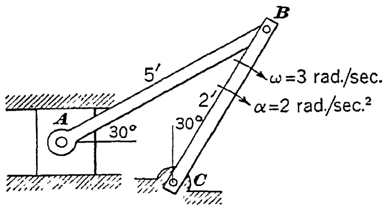


FIG. 430

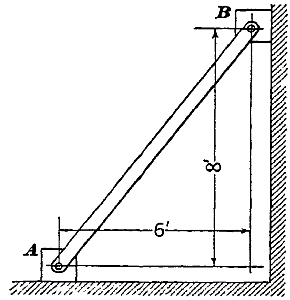


FIG. 431

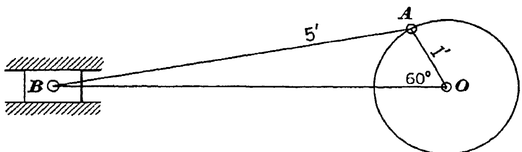


FIG. 432

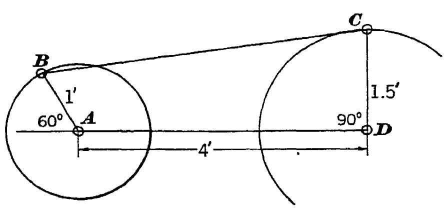


FIG. 433

CHAPTER 15

RECTILINEAR TRANSLATION OF A RIGID BODY

139. Introduction.—In Chapter 14 the motion of rigid bodies in its abstract form was studied; that is, the bodies were considered to be simply geometric forms in motion. In the present and succeeding chapters the motion of actual rigid bodies endowed with such properties as mass, weight, and momentum, and acted upon by forces external to the bodies, will be studied.

While Aristotle and Archimedes are generally credited with being the founders of Mechanics, much of their work has since been shown to have been erroneous. The real ground work of present-day dynamics was done by Galileo (1564–1642). Some historians now consider that the efforts of the Greek philosophers in the field of Mechanics were unfortunate. Many of their theories were based on unsound premises which led to false conclusions. Some of these erroneous results were accepted by the world as true for approximately 2,000 years, or until Galileo proved them false by experimentally obtained results. If the Greeks had been experimentally inclined, they probably would have devised some approximately accurate means of measuring time, as Galileo did, and would have found that the theories they arrived at by rationalization were incorrect. Accurate time-measuring instruments were first invented by Christian Huygens (1629–1695) and Robert Hooke (1635–1702).

It was not until Isaac Newton (1642–1727) published his great *Principia* (1687) that dynamics as a science was really started on its way.*

140. Newton's Laws of Motion.—Sir Isaac Newton first published his basic laws of motion in 1687. Newton formulated these laws from his study of the motion of the planets. Since the dimensions of the planets are very small when compared to the range of their motion, Newton's Laws are only applicable to the motion of a material particle.

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* *A Historical Appraisal of Mechanics*, by H. F. Girvin. International Textbook Co.

The motion of material bodies is such that, in general, they do not follow Newton's laws. The laws, however, can be applied to the study of the motion of the individual particles of the rigid body, and relationships can be deduced which definitely determine the motion of the entire body.

NEWTON'S LAWS

1. A material particle acted upon by a balanced force system receives no acceleration but remains at rest or continues to move with a uniform motion.

2. A material particle acted upon by an unbalanced force system receives an acceleration, in the direction of the resultant force, which is proportional to the resultant force and inversely proportional to the mass of the particle.

3. For every force acting on a material particle, the particle exerts an equal, opposite, and collinear force. This is what is commonly known as action and reaction.

The first law is, in reality, a special case of the second law. Since the resultant of a balanced force system is zero, the acceleration must be zero and the particle must move at a uniform rate or remain at rest.

The second law is the basic or fundamental principle of Kinetics. It states a definite relationship between force, mass, and acceleration.

141. **Mass.**—Mass has been defined in many ways, such as the quantity of matter in a body or as something which occupies space. Newton's second law presents another definition of mass. The mass of a particle is a measure of the particle's ability to resist having its state of motion changed. As previously shown,

$$M=\frac{W}{g}=\frac{\text{lb}\times\text{sec}^2}{\text{ft}}=\text{slugs}$$

If a block of wood and a block of lead of exactly the same size are placed on a smooth surface and are acted upon by equal resultant forces, experience tells us that the wood block will receive a much larger acceleration than the lead block. The lead block resists being accelerated or having its state of motion changed more than does the wood block.

This resistance to a change in the motion is generally known as inertia. Mass is a measure of the inertia possessed by a body.

142. Mathematical Statement of Newton's Second Law. Newton's second law is stated mathematically in the following manner:

$$\Sigma F = \alpha \, dm \, a \tag{1}$$

where ΣF is the resultant force acting on a small particle of mass dm, a is the acceleration which this particle receives, and α is a constant whose value varies according to the system of units which is used.

If the particle of mass dm were allowed to fall freely in a vacuum, it would be acted upon by an unbalanced resultant force dw, which is the pull of gravity on the mass dm. Experiment has shown that the particle will receive an acceleration of approximately 32.2 ft per sec per sec, or g. Equation (1) thus becomes

$$dw = \alpha \, dm \, g \tag{2}$$

If equation (2) is combined with equation (1), we obtain the following result:

$$\Sigma F = \frac{dw}{g} a \tag{3}$$

This is the equation for rectilinear motion of a particle. It will be observed that if a resultant force of 1 lb acts on a 32.2-lb particle, the particle will receive an acceleration of 1 ft per sec per sec. Thus, for convenience, engineers have adopted 32.2 lb of matter as the unit of mass. The resultant force ΣF and the acceleration a are vector quantities, but their directions are always the same. Therefore, $\frac{dw}{g}$ is a scalar quantity.

143. Transition From a Particle to a Rigid Body.—Let Fig. 434 represent any finite body which weighs W lb and is acted upon by the external forces F_1, F_2, F_3, and F_4.

This body is made up of an infinite number of particles, the mass of one of which is represented by dm. This particle is acted upon by a system of forces consisting of the weight dw and the several forces, dp_1, dp_2, dp_3, and dp_4, which represent the pressures of the surrounding particles of the body on this particular particle. The resultant of this concurrent system of forces will be a single

force, which is known as the effective force for this particle. Each and every particle of the body is acted upon by a similar force system. Since the pressures between the different particles act in pairs of equal and opposite forces (action and reaction), these pressures will cancel. The resultant of all the effective forces for all the particles of the body will be simply the resultant of the external forces which act on the body, because all the internal pressures would cancel out of any force summation which might be made.

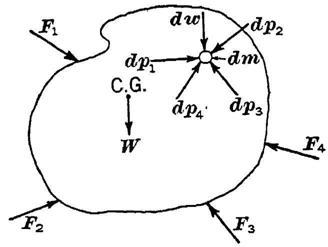

FIG. 434

Since the externally applied forces must come under one of the classifications set up in statics, the resultant of any such system of external forces must necessarily fall under one of the following cases.

1. A single force, acting through the center of gravity of the entire body. Such a force will produce rectilinear translation. Each particle of the body will receive the same acceleration.

2. A couple, in which case the motion of the body will be a rotation. Each particle of the body will travel in a circular path about some fixed axis.

3. A single force and a couple, which will cause some form of plane motion.

The idea of reducing all the effective forces acting on all the particles of a body to a single resultant effective force, a couple, or a resultant force and a couple, and of showing that this resultant effective force system is exactly equal to the resultant of the system of forces applied externally to the body was first presented by Jean le Rond D'Alembert in 1743 and is now generally known as the D'Alembert Principle.

For the case of rectilinear translation, each of the particles of the rigid body moves parallel to each of the other particles. Since every particle has the same mass *dm* and the same acceleration *a*, the effective force acting on each of the particles will have the same magnitude and the same direction. The resultant of such a system of equal and parallel effective forces would pass through the center of mass of the original body and would be parallel to and in the direction of the acceleration. As

previously stated, the resultant of the effective forces is equal to the resultant of the externally applied forces. The resultant of the effective forces for rectilinear translation must therefore be simply a single force acting through the center of mass and equal to the algebraic sum of the components of the externally applied forces in the direction of the acceleration.

If we apply Newton's Second Law to the body, we have

$$\Sigma F = \frac{W a}{g}$$

where ΣF is the algebraic sum of the components of the externally applied forces in the direction of the acceleration, and $\frac{W a}{g}$ is the resultant effective force. Also, W is the weight of the entire body, g is the acceleration of gravity, or 32.2, and a is the acceleration which the body receives.

144. **Methods of Solution.**—There are two general methods of procedure for solving problems involving translation of a rigid body.

(*a*) The resultant effective force method.

(*b*) The reversed resultant effective force method or the inertia force method.

The resultant effective force method involves a direct application of Newton's Second Law, or the relation $\Sigma F = \frac{W}{g} a$, as explained in Art. 143.

The following explanation illustrates the use of the reversed resultant effective force or inertia force method. When a body translates because it is acted upon by an unbalanced force system ΣF, it receives an acceleration and is not in equilibrium. *If a reversed resultant effective force, or inertia force,* $\frac{W}{g} a$, *which is equal to the unbalanced resultant force* ΣF, *is added to the original system of forces, this modified system of forces will be in equilibrium and the equations of statics can be applied to the body.* This is the manner in which the D'Alembert Principle is applied to the solution of problems. Some writers state the D'Alembert Principle in the following manner. The external forces acting on any body are in dynamic equilibrium with the reversed resultant effective, or inertia, force or forces.

The student must keep in mind the fact that the addition of the reversed resultant effective, or inertia, force is simply a device to aid in the solution of the free body. The only external forces which actually act on the rigid body are those which constitute the original system of external forces.

The reversed effective force, or inertia force, is the resistance of the body (inertia) to any change in its condition of motion. A wagon that is being pulled by a horse will move with an acceleration a if the force P exerted by the horse on the wagon is greater than the frictional resistance F of the wagon. If the wagon is treated as a free body, the unbalanced part of the pull of the horse is the effective force applied to the wagon. It is $P-F$, and $P-F=\frac{W}{g}a$. If a reversed effective force, or inertia force, $\frac{W}{g}a$, which is acting through the center of gravity and is directed opposite to the acceleration of the wagon, is applied to the wagon, the wagon will be in equilibrium.

Several problems will now be solved by each of these methods. The student will soon observe that there is little difference in the solutions. The reversed resultant effective force, or inertia force, method has some advantage in certain types of problems.

EXAMPLE 1

Determine the draw-bar pull required to give a 100,000-lb car an acceleration of 1 ft per sec per sec up a 1.5% grade. The total frictional resistance of the car is 500 lb

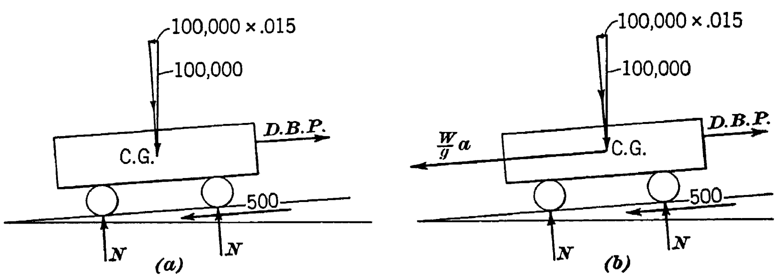

FIG. 435

Resultant Effective Force Method.—In Fig. 435 (*a*) the car is shown as a free body with all external forces acting. Since the only motion possible is along the plane, the resultant of the system,

or the resultant effective force, must be parallel to the plane. A summation parallel to the plane gives: Resultant effective force = D.B.P. − 100,000 × 0.015 − 500 = D.B.P. − 2,000.

$$\Sigma F = \frac{W}{g} a$$

$$\text{D.B.P.} - 2{,}000 = \frac{100{,}000}{32.2} \times 1$$

$$\text{D.B.P.} = 5{,}105 \text{ lb}$$

Reversed Resultant Effective Force, or Inertia Force, Method. Since the body receives an acceleration up the incline, the reversed resultant effective force, or inertia force, must act through the center of gravity of the car *opposite to the acceleration,* or down the plane. If this reversed resultant effective force, or inertia force, is added to the system of external forces, as in Fig. 435 (*b*), then the free body will be in a state of artificial equilibrium. Any of the principles of statics may now be applied to this free body. Sum forces parallel to the plane.

$$\text{D.B.P.} - 1{,}500 - 500 - \frac{100{,}000}{32.2} \times 1 = 0$$

$$\text{D.B.P.} = 5{,}105 \text{ lb}$$

The student will observe that the two methods give identical equations except for the arrangement of the terms.

EXAMPLE 2

Determine the weight W, Fig. 436 (*a*), required to cause the 1,000-lb block to move 50 ft up the plane from rest in 5 sec. Assume that $f = 0.2$.

For the 1,000-lb block,

$$s = \frac{1}{2} a t^2$$

$$50 = \frac{1}{2} a \times 25$$

$$a = 4 \text{ ft per sec per sec}$$

Because of the arrangement of the pulleys, the acceleration of W will be half that of the 1,000-lb weight, or 2 ft per sec per sec.

Resultant Effective Force Method.—Taking the 1,000-lb weight as the first free body, sum forces parallel to the plane and apply the equation

$$\Sigma F = \frac{W}{g} a$$

The resultant effective force in the direction of the motion is then given by a summation parallel to the plane.

$$T - 866 - 500 \times 0.2 = T - 966$$
$$T - 966 = \frac{1{,}000}{32.2} \times 4$$
$$T = 1{,}090.2 \text{ lb}$$

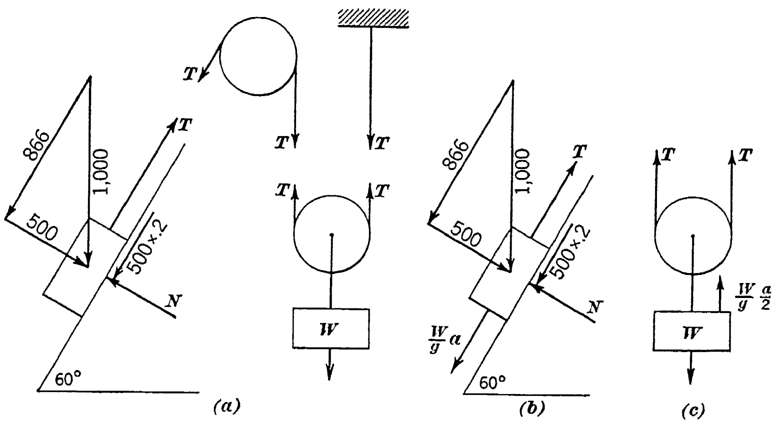

FIG. 436

Take the weight W as the second free body. The resultant force in the direction of motion is $W - 2T$.

$$\Sigma F = \frac{W}{g} a$$
$$W - 2T = \frac{W}{32.2} \times 2$$
$$W - 2 \times 1{,}090.2 = \frac{W}{32.2} \times 2$$
$$W = 2{,}324 \text{ lb}$$

Attention is again directed to the importance of correct use of signs. In this equation it is *essential that the sign of the quantity*

representing the resultant effective force be the same as that given the acceleration.

Reversed Resultant Effective Force, or Inertia Force, Method. Since the 1,000-lb weight is to be accelerated up the plane, the reversed resultant effective force, or inertia force, will act *opposite to the direction of the acceleration,* or down the plane, as indicated in Fig. 436 (*b*).

Sum forces parallel to the plane:

$$T - 866 - 500 \times 0.2 - \frac{1{,}000}{32.2} \times 4 = 0$$
$$T = 1{,}090.2 \text{ lb}$$

The weight W is accelerated downward; the reversed resultant effective force, therefore, is up. Sum forces in Fig. 436 (*c*) in a vertical direction.

$$W - 2T - \frac{W}{32.2} \times 2 = 0$$
$$W - 2 \times 1{,}090.2 - \frac{W}{32.2} \times 2 = 0$$
$$W = 2{,}324 \text{ lb}$$

Again the two methods give identical equations, except for the arrangement of the terms.

PROBLEMS

584. A body weighing 1,000 lb rests on a horizontal plane. A force P directed 30° above the horizontal is acting on the body. If the body attains a velocity of 15 ft per sec in 5 sec and $f = 0.2$, what is the magnitude of P? *Ans. 303.5 lb.*

585. A constant force of 500 lb acts on a car, changing its velocity from 45 mi per hr to 15 mi per hr in 30 sec. What is the weight of the car, and how far does it travel during the 30 sec?

586. A car weighing 3,000 lb starts up a 15° incline at 45 mi per hr. The total frictional resistance of the car is 100 lb. (*a*) How long will the car continue to move up the incline? (*b*) How far will it go? (*c*) If, after coming to rest, it starts back down the incline, with what velocity will it reach the bottom?

587. What draw-bar pull is required to change the speed of a 100,000-lb car from 15 mi per hr to 30 mi per hr in a half-mile, while the car is going up a 2% grade? Car resistance is 10 lb per ton.

588. If $f = 0.3$, what weight W is required to give the 500-lb weight, Fig. 437, a velocity of 20 ft per sec after moving 50 ft up the plane? *Ans. 503 lb.*

589. If $f = 0.3$, for both planes, Fig. 438, what are the tension in the cord and the time required to move 12 ft from rest?

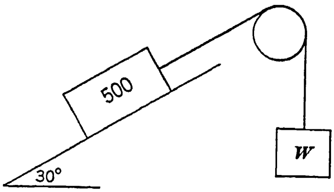

FIG. 437

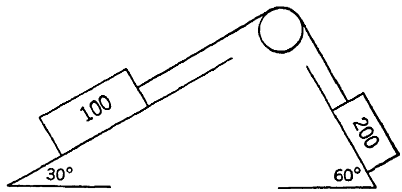

FIG. 438

590. If the cable which supports a certain elevator has a safety factor of 2, what maximum acceleration upward can the elevator car receive?

591. If the wheels of an automobile are locked by the brakes and the car slides 36 ft and stops in 3 sec, what is the coefficient of friction for the tires and road? Assume that the car is decelerated at a constant rate.

145. Kinetic Reactions During Translation.—From study of the examples and problems of Art. 144, the student has observed that, where a summation parallel to the line of motion will solve the problem, there is no choice between the two methods of solution illustrated.

In problems involving kinetic reactions, the reversed resultant effective force method, or inertia force method, has a decided advantage.

Kinetic reactions are forces present only when the body is receiving an acceleration. When an automobile stands still or moves at a constant rate of speed in a straight line, the proportion of the weight of the car which is carried by each wheel is determined entirely by the position of the center of gravity of the car. If the car is caused to speed up, the weight carried by the front wheels is decreased and that carried by the rear wheels is increased. When the car is slowing down, the weight on the front wheels increases and that on the rear wheels decreases. The portions of the reactions which are due to the body receiving an acceleration are known as the kinetic reactions.

EXAMPLE

A 3,600-lb automobile, Fig. 439 (a), is traveling 60 miles an hour when the brakes are applied. If $f = 0.6$ for the tires and road, what is the shortest distance in which the car can be stopped? What are the front-wheel and rear-wheel reactions while the car is stopping?

Reversed Resultant Effective Force, or Inertia Force, Method. The car tends to continue in motion at 60 mi per hr, or to resist

reduction of its speed. Hence, the reversed resultant effective force, or inertia force, acts forward through the center of mass, or *in the opposite direction to the acceleration.* If this reversed resultant effective force, or inertia force, is added to the free body in Fig. 439 (*a*), an artificial state of equilibrium will be established. The force system on this free body can be solved by the methods of statics.

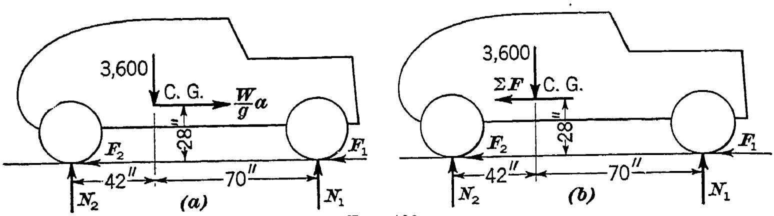

FIG. 439

Examination of the free body shows there are three unknown quantities, a, N_1, and N_2. Three independent equations are required for a solution. They are: $\Sigma H=0$, $\Sigma V=0$, and $\Sigma M=0$.

$$\Sigma H=0$$

$$-F_1-F_2+\frac{W\,a}{g}=0$$

Since $F_1+F_2=3{,}600\times 0.6=2{,}160$,

$$-2{,}160+\frac{3{,}600\,a}{32.2}=0 \tag{1}$$

$$a=19.3 \text{ ft per sec}^2$$

$$\Sigma M_{N_1}=0$$

$$-112\,N_2+3{,}600\times 70-\frac{3{,}600\times 19.3}{32.2}\times 28=0 \tag{2}$$

$$N_2=1{,}710 \text{ lb}$$

$$\Sigma V=0$$

$$N_1+1{,}710-3{,}600=0 \tag{3}$$

$$N_1=1{,}890 \text{ lb}$$

$$v^2=v_0^2+2\,a\,s$$

$$0=88^2+2(-19.3)s$$

$$s=200.6 \text{ ft}$$

Resultant Effective Force Method.—Since the free body in Fig. 439 (*b*) is not in equilibrium, but is being acted upon by a resultant force, the ordinary methods of statics do not apply. However, the following principles do apply:

(*a*) A resultant force is equal to the algebraic sum of its component forces.

(*b*) The moment of a resultant force with respect to any axis is equal to the algebraic sum of the moments of the component forces.

By Art. 143, the resultant effective force for the free body is $\Sigma F = \frac{W}{g} a$. It acts through the center of gravity in the direction of the acceleration, as indicated in Fig. 439 (*b*).

Summing forces in the direction of ΣF according to (*a*) gives:

$$\Sigma F = F_1 + F_2 = 3{,}600 \times 0.6 = 2{,}160$$

$$2{,}160 = \frac{3{,}600\ a}{32.2} \tag{1}$$

$$a = 19.3 \text{ ft per sec}^2$$

According to (*b*) the moment of ΣF with respect to any axis must be equal to the sum of the moments of all the component forces with respect to the same axis. If an axis through the intersection of F_1 and N_1 is selected,

$$2{,}160 \times 28 = \frac{3{,}600 \times 19.3}{32.2} \times 28 = -112\ N_2 + 3{,}600 \times 70 \tag{2}$$

$$N_2 = 1{,}710 \text{ lb}$$

Since ΣF has no component in the vertical direction,

$$N_1 + 1{,}710 - 3{,}600 = 0$$

$$N_1 = 1{,}890 \text{ lb}$$

$$v^2 = v_0^2 + 2\ a\ s$$

$$0 = 88^2 + 2(-19.3)s$$

$$s = 200.6 \text{ ft}$$

PROBLEMS

592. If the car in the preceding example has brakes on the rear wheels only, how far will it travel while coming to rest? What are the wheel reactions? *Ans. 369 ft;* $N_1 = 1{,}640$ *lb;* $N_2 = 1{,}960$ *lb.*

593. A homogeneous block, Fig. 440, which is 2 ft×2 ft×8 ft and weighs 400 lb, is attached to the car at *A* by a hinge. The car weighs 1,000 lb. Determine the maximum force *P* that may be applied to the car without overturning the block and the amount and direction of the hinge reaction.

594. Determine the wheel reactions R_1 and R_2 for the car in Problem 593 when the force *P* has its maximum value. The center of gravity of the car is midway between the wheels and 12 in. above the rails.

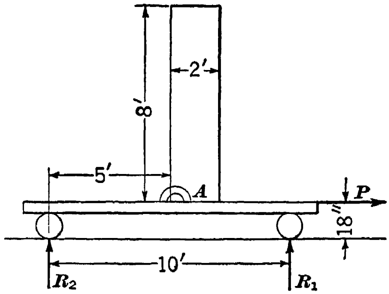

FIG. 440

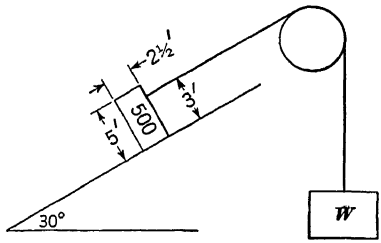

FIG. 441

595. If $f=0.3$ for the 500-lb weight on the plane in Fig. 441, what maximum weight W can be attached to the cord without overturning the 500-lb weight? What are the tension in the cord and the acceleration?

596. If the block in Fig. 440 is not attached to the car and is free to slide on the car with $f=0.3$ for the car and block, what is the minimum time in which the car can be brought to rest from a speed of 30 mi per hr without disturbing the block? If the wheels are not to slide, what is the minimum value of f for the wheels and track? There are brakes on all wheels.

146. Translation With a Variable Acceleration.—During many translations the acceleration and therefore the effective force, and also the reversed effective force, or inertia force, are variable quantities and *must* be expressed in terms of s, t, or v. Such problems require the use of the basic differential equations of translation, which are $v=\frac{ds}{dt}$, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$, and $v\,dv=a\,ds$; and the methods of solution can best be illustrated by examples.

EXAMPLE 1

Fig. 442 (*a*) represents two weights of 100 lb and 10 lb connected by a 20-ft cable or chain which weighs 2 lb per ft and passes over a pulley. The frictional resistance between the 100-lb weight and a horizontal plane is 8 lb. Determine the velocity of the weights after the 10-lb weight has fallen 20 ft from the pulley.

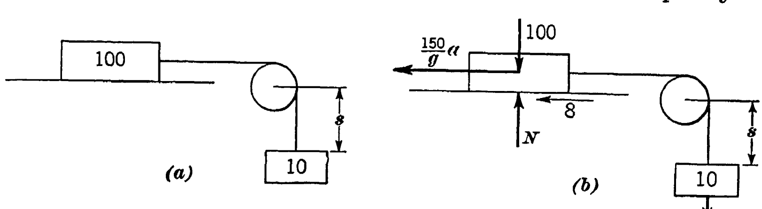

FIG. 442

Resultant Effective Force Method:

$$\Sigma F = 10 + 2s - 8 = 2 + 2s$$

The weight of the entire system is 150 lb.

$$2 + 2s = \frac{150}{32.2}\, a$$
$$a = 0.43 + 0.43s \text{ and } v\,dv = a\,ds$$
$$\int^{v} v\,dv = \int_0^{20} (0.43 + 0.43s)\,ds$$
$$v = 13.72 \text{ ft per sec}$$

Reversed Resultant Effective Force, or Inertia Force, Method. Since the acceleration is to the right, the reversed resultant effective force, or inertia force, acts to the left, as indicated in Fig. 442 (*b*).

Summing forces along the line of motion, we obtain:

$$10 + 2s - 8 - \frac{150}{32.2}\, a = 0$$
$$a = 0.43 + 0.43s$$

Solve by integration as in the previous method.

EXAMPLE 2

A motor boat which weighs 1,500 lb has the power shut off when its speed is 30 mi per hr. The speed drops to 15 mi per hr in 30 sec. Assuming that the resistance offered by the water is $K\,v$, determine the value of K.

The resultant effective force is $K\,v$.

$$K\,v = \frac{1{,}500}{32.2}\, a$$
$$a = \frac{32.2}{1{,}500}\, K\,v$$
$$a = \frac{dv}{dt} = \frac{32.2}{1{,}500}\, K\,v$$
$$\int_{22}^{44} \frac{dv}{v} = \frac{32.2}{1{,}500}\, K \int_0^{30} dt$$
$$\log_e 2 = 0.645\, K$$
$$K = 1.08$$

EXAMPLE 3

A 200-lb block rests on a plane which is inclined at 30° with the horizontal, as indicated in Fig. 443. The block is pushed down the plane a distance of 6 in. against a coil spring whose scale is 5,000 lb per in. (a force of 5,000 lb is required to compress the spring 1 in.). The block is then released and it is pushed up the plane by the spring. The spring acts on the block only for the 6 in. during which it was compressed. (*a*) If $f=0.3$ for the plane, with what velocity does the block pass the 6-in. point? (*b*) How far up the plane does it go?

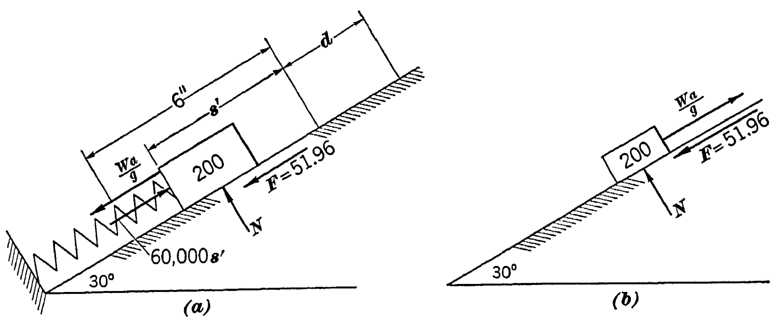

FIG. 443

Reversed Resultant Effective Force, or Inertia Force, Method. The force required to compress the spring 1 ft $=5{,}000\times12=60{,}000$ lb.

$$\Sigma F \text{ parallel to the plane in Fig. 443 } (a)=0$$

$$100+51.96+\frac{200\,a}{32.2}-60{,}000\,s'=0$$

$$a=9{,}660\,s'-24.4$$

The acceleration is opposite to the initial motion, and the displacement therefore must have the negative sign.

$$\int_0^v v\,dv=\int -a\,ds=\int_{0.5}^{0} -(9{,}660\,s'-24.4)\,ds$$

$$v^2=2{,}390$$

$$v=48.9 \text{ ft per sec}$$

When the block passes the 6-in. point and is free of the spring, the block in Fig. 443 (*b*) is the free body.

ΣF parallel to the plane in Fig. 443 $(b) = 0$

$$100 + 51.96 - \frac{200\,a}{32.2} = 0$$

$$a = 24.4 \text{ ft per sec}^2$$

$$v^2 = v_0^2 + 2\,a\,s$$

$$0 = 2{,}390 + 2(-24.4)\,d$$

$$d = 48.97 \text{ ft}$$

PROBLEMS

597. A cable hangs over a pulley with 20 ft on one side and 25 ft on the other side at the instant it is released. If the cable weighs 1 lb per ft and the pulley offers a constant frictional resistance of 3 lb, with what velocity will the end of the cable leave the pulley? How long will it take for the cable to fall free of the pulley? *Ans. 25.1 ft per sec; 3.13 sec.*

598. A chain, 10 ft long, is stretched out on a 30° inclined plane with the lower end of the chain at the edge of the plane. If $f = 0.2$ and the chain weighs 3 lb per ft, with what velocity will the chain leave the plane?

599. A man jumps from a stationary balloon. If he attains a velocity of 60 ft per sec before his parachute becomes effective, and the air resistance is assumed to be $\frac{W\,v^2}{256}$, with what terminal velocity will the man reach the ground when jumping from a great height?

600. Let the plane in Example 3, Art. 146, be smooth and the 200-lb block be attached to the spring. The block is displaced 3 in. downward from its position of rest or equilibrium on the plane and then it is released. With what velocity will the block first pass the equilibrium position?

REVIEW PROBLEMS

601. Determine the horizontal force P required to give the 500-lb weight of Fig. 444 a velocity of 10 ft per sec, after the block has moved 30 ft up the plane. Assume that $f = 0.2$. *Ans. 473 lb.*

602. An elevator starts from rest and attains an upward velocity of 10 ft per sec after moving 20 ft. If the acceleration of the elevator is constant, what pressure will a 175-lb man exert on the floor of the elevator? If the elevator is then decelerated at the same rate, what is the pressure?

603. Pulley A, Fig. 445, is free to move; pulley B is fixed. The cord passing over A is fixed at C. Determine: (a) the acceleration of each weight; (b) the tension in each cord; and (c) the distance traveled by the 20-lb weight in 2 sec.

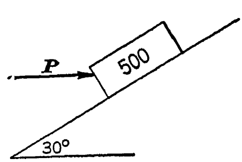

FIG. 444

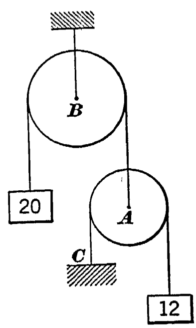

FIG. 445

604. Find the tension in the supporting cord A, Fig. 446, and the velocity of each weight 3 sec after starting from rest.

605. What weight will remain stationary at B if the weights C and D are as shown in Fig. 446?

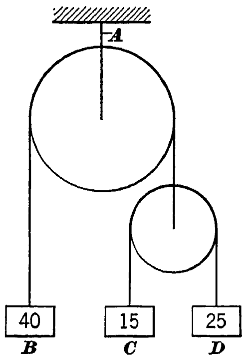

FIG. 446

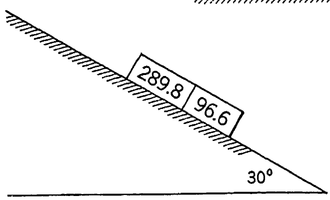

FIG. 447

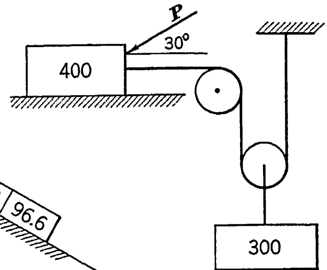

FIG. 448

606. A freight car has a speed of 10 mi per hr when it is switched up a 1% grade. The car resistance is 8 lb per ton. How far up the grade will the car go?

607. A train of 30 cars, each weighing 40 tons, starts up a 1% grade at 30 mi per hr. Frictional resistance is 8 lb per ton. If the drawbar pull is 35,000 lb, with what speed will the train pass a point 2 mi up the grade?

608. A body slides down a plane that is inclined 30° with the horizontal and for which $f = 0.4$. Determine the time required for the body to move 40 ft from rest. What angle of inclination of the plane will be required for a constant speed of the body down the plane?

609. If $f = 0.5$ for the 96.6-lb block in Fig. 447 and $f = 0.3$ for the 289.8-lb block, what is the pressure of one block against the other?

610. The 300-lb weight in Fig. 448 has an initial downward velocity of 20 ft per sec. What force P will be required to bring the weights to rest after the 400-lb weight has traveled 60 ft? For the horizontal plane the coefficient $f = 0.2$.

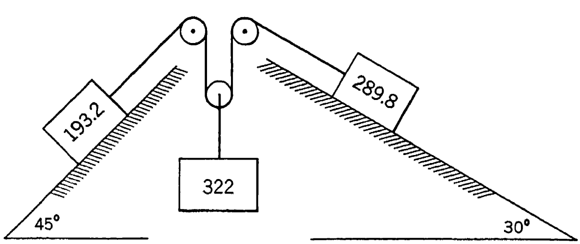

FIG. 449

611. A traveling smelter crane carries a 40-ton ladle suspended by cables. The distance from the point of tangency of the cables and the cable drum to

the center of gravity of the ladle is 40 ft. The crane attains a speed of 10 ft per sec in a distance of 50 ft after starting from rest. Determine the total load on the cables and the distance the ladle lags behind the point of tangency of the cables and the drum.

612. Solve for the tension in the rope, Fig. 449, and the distance moved by the 322-lb weight in 2 sec from rest. Both planes are frictionless.

613. Wheel B, Fig. 450, does not turn, but it slides on the track and $f=0.3$. If $P=200$ lb, what are the vertical reactions at A and B?

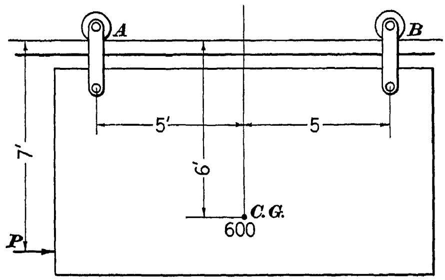

FIG. 450

614. Determine the minimum weight W in Problem 595 for motion down the 30° plane without tipping.

615. Determine the time for the 1,000-lb weight, Fig. 451, to move 30 ft. Also compute the tension in the rope attached to the 325-lb weight. Assume that $f=0.2$.

616. A 4,000-lb automobile with four-wheel brakes has its center of gravity 26 in. above the ground and its wheel base is 120 in. The center of gravity is 50 in. in front of the rear wheels. If the speed is reduced from 60 mi per hr to 15 mi per hr in 40 sec, what are the reactions at the front and rear wheels?

617. If the car in Problem 616 has its brakes adjusted so that the maximum braking effect will be developed at all wheels and $f=1$, what is the shortest distance in which it can be stopped? What are the wheel reactions while stopping?

618. A 100,000-lb car and an 80,000-lb car are coupled together with the heavy car in front. The speed of the cars is reduced from 30 to 15 mi per hr while they move 500 ft down a 1% grade. If the braking effect on the rear car is 50 per cent greater than that on the front car, what is the tension in the coupler? *Ans. 1,545 lb.*

619. The car in Fig. 452 is brought to rest in a distance of 18 in. from a speed of 5 mi per hr by a constant resisting force acting at the coupler. Compute this resisting force and the reactions at the wheels.

620. If in Problem 619 a coil-spring bumper acts against the coupler and the resistance offered by the spring is proportional to the displacement, what is the scale of the spring?

621. A 500-lb block slides down a 30° plane 100 ft long and then on to a horizontal plane. If the block starts from rest and $f=0.3$, how far will it move? *Ans. 180 ft.*

622. In Fig. 453 is illustrated a method for taking street cars up steep hills. If the rolling resistance of each car is 20 lb per ton and the brakes on the street car are defective, what maximum value can P have? Neglect the effect of the acceleration normal to the planes.

623. Determine the maximum force P which may be applied to the car in Fig. 454 without causing the 500-lb weight to tip. What acceleration will the car receive?

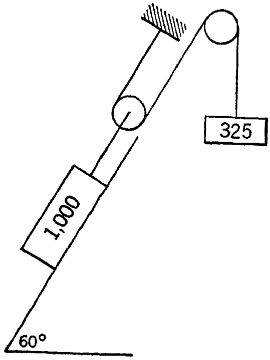

FIG. 451

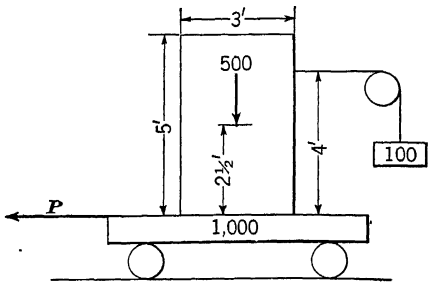

FIG. 454

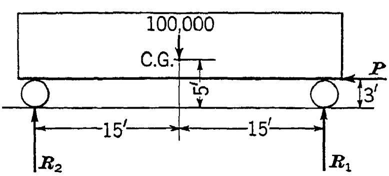

FIG. 452

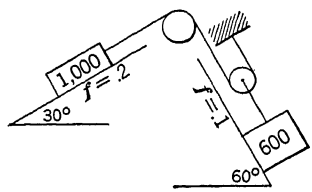

FIG. 455

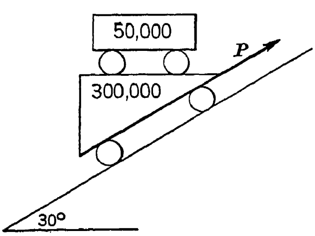

FIG. 453

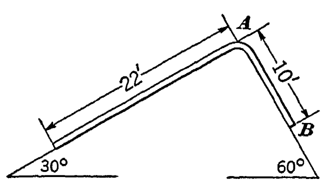

FIG. 456

624. Find the time required for the 1,000-lb block in Fig. 455 to move 20 ft from rest. Also find the tension in the rope.

625. If the resistance offered by the air to a certain ball is $\frac{W v^2}{10{,}000}$ and the ball is thrown straight up with an initial velocity of 300 ft per sec, how high will the ball go? How long will it take to reach this maximum elevation? *Ans. 357 ft; 3.87 sec.*

626. A 100-lb body slides down a 30° plane for which $f=0.2$. If the resistance offered by the air is $0.5\,v$ and the body has an initial velocity of 10 ft per sec, what maximum velocity can it attain? How long will it take to reach a velocity of 30 ft per sec?

627. A cable which weighs 1 lb per ft is placed on two smooth planes, as in Fig. 456. Determine the velocity with which the end B will pass point A.

628. A small boat which weighs 500 lb is moving with a velocity of 10 ft per sec when the power is shut off. If the resistance offered by the water is $2v$, how far will the boat move before it comes to rest? When will it have a velocity of 5 ft per sec?

629. Assume that the resistance of the air varies as the square of the velocity and that a parachute falling with a velocity of 30 ft per sec has a resistance of 2 lb per sq ft. Determine the diameter which the parachute must have if a man and parachute weighing 190 lb are to descend with a speed not to exceed 20 ft per sec. *Ans. 16.5 ft.*

630. A 120,000-lb railroad car has a speed of 5 mi per hr when it strikes a bumping post at a point 4 ft above the top of the rails. The springs in the bumping post have a scale of 30,000 lb per in. (*a*) How far are the springs compressed? (*b*) What maximum acceleration does the car receive?

631. A 20-lb weight falls freely for 5 ft and then strikes a coil spring which has a scale of 50 lb per in. What is the maximum compression of the spring?

632. Determine the distance the man in Problem 599 falls after the parachute opens and before he reaches the terminal velocity of 16 ft per sec.

633. A 96.6-lb body is pulled up a smooth 30° plane by a force $F=6t^2$ acting parallel to the plane. If the body has a velocity of 5 ft per sec up the plane when the force $F=6t^2$ begins to act, what are the velocities of the body 0.3 sec and 1 sec later?

CHAPTER 16

CURVILINEAR MOTION

147. Acceleration During Curvilinear Motion.—In Arts. 121, 122, and 123, the kinematic motion of a particle along any plane curve was studied.

It was found that the velocity of the particle at any point on the curve was in the direction of the tangent to the curve and was given by $v_t = \rho\,\omega$, where ρ is the instantaneous radius of curvature and ω is the instantaneous angular velocity of the radius vector from the particle to the center of curvature.

The particle may or may not have an acceleration in the direction of the tangent, but must have an acceleration along the radius of curvature, or normal to the tangent. This acceleration is given by

$$a_n = \rho\,\omega^2 = \frac{v_t^2}{\rho}$$

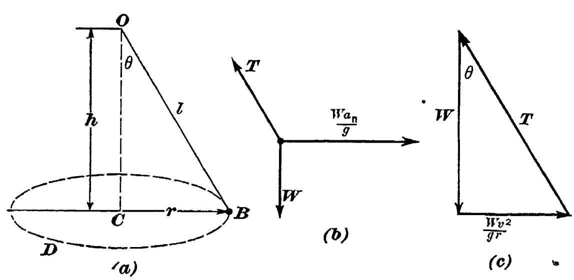

FIG. 457

148. Conical Pendulum.—Let Fig. 457 (*a*) represent a conical pendulum, consisting of a small ball or weight *B* suspended from *O* by a weightless cord. The ball rotates about the line *OC* in such a manner that the ball travels in a circular path in the horizontal plane *BCD*, and the cord *BO* generates the surface of a cone.

For any given value of θ the ball will have a constant speed in its circular path, if it is assumed that all frictional forces are

neglected. Thus, $a_t=0$ and the tangential resultant effective force is

$$\Sigma F_t=\frac{W}{g}a_t=0$$

In the vertical plane through O, B, and C, the two forces T and W act on the ball. The resultant of these forces is the resultant effective force

$$\Sigma F_n=\frac{W}{g}a_n$$

If the ball is to travel in a circular path about C, this resultant effective force must act along the radius BC, or normal to the axis CO, and toward C.

In Fig. 457 (b) the ball is shown as a free body, with the reversed resultant effective force, or the inertia force, $\frac{W}{g}a_n$ or $\frac{W}{g}\frac{v^2}{r}$, added. This free body is in equilibrium.

Therefore, $\Sigma H=0$, $\Sigma V=0$, and $\Sigma M=0$.

$$\Sigma M_O=0$$

$$\frac{W}{g}\frac{v^2}{r}h-W\,r=0$$

$$v=r\sqrt{\frac{g}{h}}$$

If N is the number of revolutions per second,

$$N=\frac{v}{2\pi\,r}$$

$$N=\frac{r\sqrt{\frac{g}{h}}}{2\pi\,r}=\frac{1}{2\pi}\sqrt{\frac{g}{h}}$$

This relationship shows that the height h is inversely proportional to the square of the speed of rotation.

PROBLEMS

634. A 10-lb ball is attached to a cord 6 ft long and is revolving about a vertical axis so that the cord makes an angle of 45° with the axis. Determine the rpm and the tension in the cord. *Ans. 26.3 rpm; 14.14 lb.*

635. Show that, if the cord in Problem 634 is 8 ft long, the distance h in Fig. 457 (a) will be unchanged when both pendulums have the same rpm.

636. A 5-lb ball is attached to a 5-ft cord as in Fig. 457. If the linear speed of the ball is 10 ft per sec, how high will the ball rise?

637. If ω is constant at 120 rpm, what stress will be developed in the cord AB in Fig. 458?

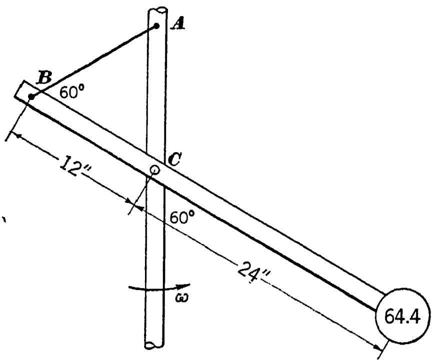

FIG. 458

149. Superelevation of Rails and Banking of Highways. When a railway car or an automobile moves around a curve on a level track or highway, there is a tendency to skid to the outside of the curve. This skidding is prevented in the case of the railway car by the pressure of the tracks against the wheel flanges; and for the automobile by the friction at the road surface.

This skidding action is due to the resistance offered by the car to a change in the direction of its motion. While rounding a curve, the car is constantly forced to the inside of the curve by the pressure of the rails against the wheel flanges or by a frictional force at the road surface.

Fig. 459 (a) is the free-body diagram for a railway car going around a flat curve. The center of curvature is in the axis YY, at a distance r from the center of gravity of the car. The car is being forced toward the center of curvature by the pressure P of the rails on the flanges. The reversed resultant effective force, or the inertia force, $\frac{W}{g}\frac{v^2}{r}$ *acts normally to the axis YY through the center of gravity of the car.* Examination of the free body shows that the reversed resultant effective force or inertia force and P form a couple, which causes the rail reactions R_1 and R_2 to become unequal. To overcome this condition the outer rail is elevated.

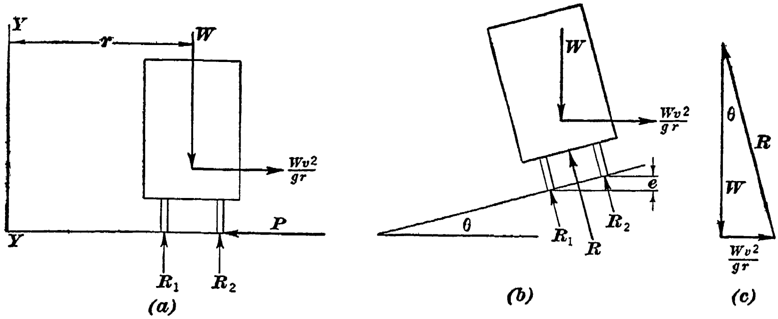

FIG. 459

In Fig. 459 (*b*) the outer rail is elevated a distance e such that $R_1 = R_2$. Their resultant R acts midway between the rails and through the center of gravity of the car. The car is held in equilibrium by the three forces R, W, and $\frac{W\,v^2}{g\,r}$. Fig. 459 (*c*) is the force triangle for this free body. In this triangle,

$$\tan\theta = \frac{v^2}{g\,r}$$

If the distance between the centers of the rails is G, the super-elevation of the outside rail is given by

$$e = G\sin\theta = G\tan\theta \text{ (when } \theta \text{ is small)}$$

$$e = \frac{G\,v^2}{g\,r}$$

For the case of an automobile on a highway curve, if the wheel reactions are equal there is no tendency to skid and

$$\tan\theta = \frac{v^2}{g\,r}$$

If the wheel reactions are not equal, there is a tendency to skid either to the inside or to the outside of the curve, the direction depending on the speed of the car. When skidding is impending, friction opposes the skidding and the car is held in equilibrium by the action of $\frac{W\,v^2}{g\,r}$, W, and the resultant reaction R of the plane, as indicated in Fig. 460 (*a*).

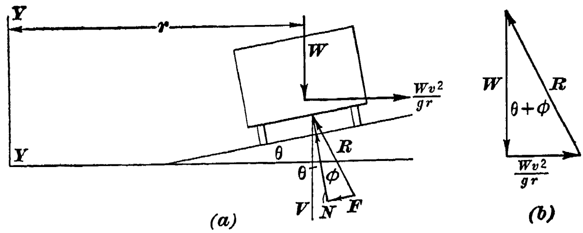

FIG. 460

Fig. 460 (*b*) is the force triangle for this case.

$$\tan(\theta+\phi)=\frac{v^2}{g\,r}$$

EXAMPLE 1

An 80,000-lb railway car goes around a curve of 3,000-ft radius at 45 mi per hr. Determine the superelevation of the outer rail which will be required to produce equal reactions at the rails. The distance between the centers of the rails is 4.9 ft. What will be the flange pressure if the car goes around the curve at 60 mi per hr? Fig. 461 is the free-body diagram.

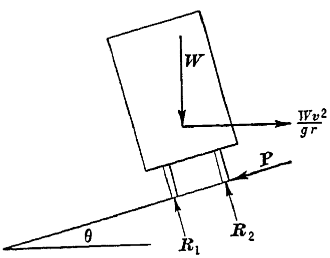

FIG. 461

$$e=\frac{G\,v^2}{g\,r}=\frac{4.9\times 66^2}{32.2\times 3{,}000}=0.221\text{ ft}$$

$$e=0.221\times 12=2.65\text{ in.}$$

$$\tan\theta=\frac{v^2}{g\,r}=\frac{66^2}{32.2\times 3{,}000}=0.045$$

$$\theta=2.6^\circ\text{ and }e=4.9\times 0.045=0.221\text{ ft}$$

In the free-body diagram shown in Fig. 461, a summation of forces parallel to the plane of the rails gives the following equation:

$$P+80{,}000\sin 2.6^\circ-\frac{80{,}000}{32.2}\times\frac{88^2}{3{,}000}\cos 2.6^\circ=0$$

$$P=6{,}390-3{,}600=2{,}790\text{ lb}$$

EXAMPLE 2

At what maximum speed can an automobile go around a track of 1,000-ft radius without skidding if the track is banked 15° and $f=0.6$?

$$\tan(15°+31°)=\frac{v^2}{32.2\times 1{,}000}$$

$$v^2=1.0355\times 32.2\times 1{,}000=33{,}343$$

$$v=182.5 \text{ ft per sec}=124.5 \text{ mi per hr}$$

PROBLEMS

638. An 80,000-lb railroad car, with its center of gravity 5 ft above the tracks, goes around a flat curve of 1,000-ft radius. If the distance between the centers of the rails is 4.9 ft, what speed will produce impending tipping? *Ans. 85.7 mi per hr.*

639. If the speed of the car in Example 1 is reduced to 6 mi per hr, what is the pressure on the wheel flanges?

640. What is the safe maximum speed without tipping or skidding for a 3,000-lb automobile around a flat curve of 250-ft radius? The center of gravity is 26 in. above the road and the wheel tread is 58 in., and $f=0.6$.

641. A highway curve of 400-ft radius is banked to give equal wheel reactions for a speed of 30 mi per hr. If $f=0.6$, what is the maximum safe speed which may be attained without skidding?

642. A 100,000-lb railroad car, which has its center of gravity 50 in. above the tops of the rails, goes around an 800-ft radius curve. The outer rail is 4 in. above the inner rail. What side thrust does the car exert on the rails when the speed is 45 mi per hr? Is this a safe speed for the car, if the distance between the centers of the rails is 4.9 ft? *Ans. 10,070 lb.*

643. Determine the pressure of the rails against the wheel flanges if a 100,000-lb car goes around an 8° curve at a speed of 30 mi per hr (an 8° curve is one for which a 100-ft chord subtends an 8° angle at the center). The curve was designed for a speed of 15 mi per hr, and the distance between the centers of the rails is 4.9 ft.

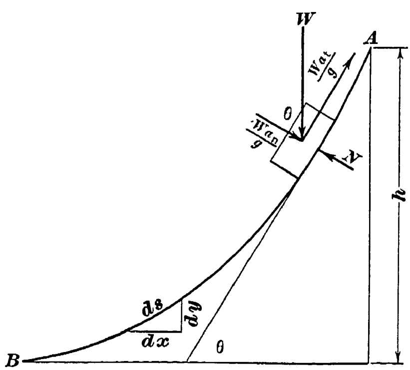

FIG. 462

150. Motion on a Smooth Vertical Curve.—Let W represent any body sliding down a smooth curve in a vertical plane from A to B, Fig. 462. The only forces acting on the body are the pull of gravity and the normal reaction of the plane.

Fig. 462 shows the free body with the inertia forces added.

A force summation in the tangential direction gives the following equation:

$$W \sin \theta - \frac{W}{g} a_t = 0$$

$$a_t = g \sin \theta$$

$$v\,dv = a\,ds = g \sin \theta\,ds$$

θ varies with the position on the curve, but $\sin \theta = \frac{dy}{ds}$.

$$\int_{v_0}^{v} v\,dv = \int_0^h g\,dy$$

$$v^2 = v_0^2 + 2\,g\,h$$

This equation shows that, when a body slides down a smooth curve with no forces acting but gravity and the reaction of the surface, *the body will attain the same speed that it would have attained if the body had fallen freely through the same vertical distance.*

If the body moves up a smooth curve in a vertical plane, it will lose speed according to the same law.

EXAMPLE 1

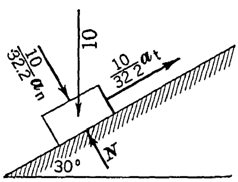

FIG. 463

A 10-lb weight slides down a smooth 30° plane for 50 ft. What maximum speed will it attain, and what time will be required to travel the 50 ft? How long would it take for the body to fall freely through the vertical distance of 25 ft?

$$\Sigma F \text{ parallel to the plane in Fig. } 463 = 0$$

$$10 \sin 30° - \frac{10}{32.2} a_t = 0$$

$$a_t = 16.1 \text{ ft per sec per sec}$$

$$s = \frac{1}{2} a\,t^2$$

$$t^2 = \frac{2 \times 50}{16.1} = 6.22$$

The time required to slide is

$$t = 2.49 \text{ sec}$$

Also,

$$v^2 = v_0^2 + 2\,g\,h$$

$$v^2 = 0 + 2 \times 32.2 \times 50 \times 0.5$$

$$v = 40.1 \text{ ft per sec}$$

For free falling,

$$v^2 = v_0^2 + 2\,g\,s$$
$$v^2 = 0 + 2 \times 32.2 \times 25$$
$$v = 40.1 \text{ ft per sec}$$
$$s = \frac{1}{2}\,g\,t^2$$
$$25 = \frac{1}{2} \times 32.2\,t^2$$

The time required for the fall is

$$t = 1.245 \text{ sec}$$

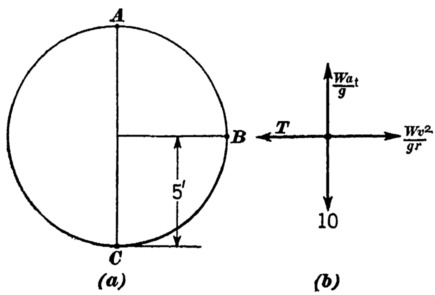

Fig. 464

EXAMPLE 2

A 10-lb weight is attached to a 5-ft cord and is rotating in a vertical circle, Fig. 464 (*a*). If the weight has a velocity of 15 ft per sec when it passes the point *A*, what is its velocity at *C*? What is the tension in the cord at point *B*?

$$v_C^2 = v_A^2 + 2\,g\,h$$
$$v_C^2 = 225 + 2 \times 32.2 \times 10$$
$$v_C = 29.4 \text{ ft per sec}$$
$$v_B^2 = 225 + 2 \times 32.2 \times 5$$
$$v_B = 23.4 \text{ ft per sec}$$

When the weight is at *B*, its velocity is 23.4 ft per sec in the vertical direction. Since the speed with which the weight moves along the curve is changing, the weight has both a tangential acceleration and a normal acceleration. Fig. 464 (*b*) is the free-body diagram for the weight when it is at the point *B*. Because the weight has both a tangential acceleration and a normal

acceleration, it has two reversed resultant effective forces, or inertia forces, which are shown in the figure. The free body is in equilibrium.

The unknown tangential reversed resultant effective force or inertia force is eliminated by summing forces in the direction of T.

$$T=\frac{10}{32.2}\times\frac{23.4^2}{5}=34 \text{ lb}$$

PROBLEMS

644. A freight train is traveling 45 mi per hr along a straight track. If a ball is thrown horizontally forward at an angle of 30° with the direction of the track and with a velocity of 90 ft per sec relative to the train, with what speed will it strike the ground 15 ft below its starting point? Air resistance may be neglected.

645. A weight is revolved in a vertical plane at the end of a rope L feet long. What minimum tangential velocity in feet per second, at the lowest point in the path, will just permit the weight to follow the circular path at the top of the circle? *Ans.* $v=\sqrt{5\,g\,L}$.

646. A 20-lb weight, attached to the end of a 15-ft rope, is revolved at minimum speed in a vertical plane. If the rope is just strong enough to have a factor of safety of 3, what is the breaking strength of the rope?

647. Determine the height h required to cause a 1,000-lb car to exert a 800-lb pressure on the rails, as the car passes the point A on the "loop the loop" in Fig. 465, all friction and air resistances being neglected.

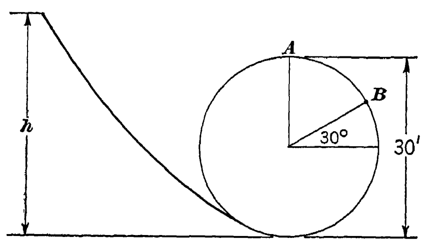

FIG. 465

648. What is the track pressure for the car in Problem 647 when it passes point B?

649. A 150-lb man is swinging on a swing with ropes 20 ft long. If the man passes the lowest point with a velocity of 30 ft per sec, how high will he rise? What would happen if he should attain a maximum height of 22 ft above the lowest point? *Ans. 13.95 ft.*

650. A 5-lb weight starts from point A on a smooth cylinder, Fig. 466, and slides to B, where it leaves the surface of the cylinder and strikes the ground at C. Determine: (a) the initial velocity at A, (b) the striking velocity at C, and (c) the distance x.

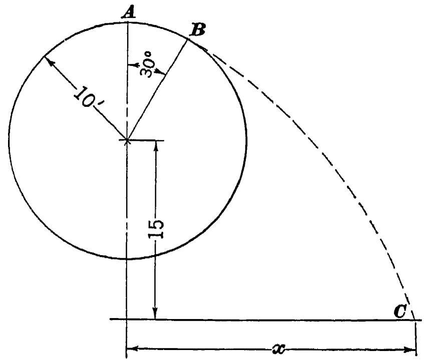

FIG. 466

REVIEW PROBLEMS

651. A 5-lb weight is attached to a 10-ft cord. The weight makes 30 rpm about a vertical axis through the point of attachment of the cord. What angle will the cord make with the axis? What is the tension in the cord? *Ans. 70.95°; 15.33 lb.*

652. A hollow sphere 12 ft in diameter is making 40 rpm about a vertical diameter. If a marble is dropped into the sphere, what position will the marble assume after equilibrium has been established?

653. Fig. 467 represents a type of revolving swing seen in amusement parks. How many rpm will be required to cause the 500-lb car to assume the position indicated?

654. An airplane which weighs 20,000 lb is traveling 250 mi per hr. It banks at a 45° angle with the horizontal when turning. Determine: (*a*) the radius of curvature of its path and (*b*) the lift force (force perpendicular to a chord connecting the wing tips and the longitudinal axis of the plane).

655. At what angle should an airplane bank to make a turn of 600-ft radius, if the speed of the plane is 500 mi per hr and there is no side slip of the plane? Is this possible?

656. If the pilot in Problem 654 weighs 165 lb, what pressure does he exert normal to the plane seat?

657. A motorcycle track, 100 ft in diameter, has its sides banked 75° with the horizontal. What is the required speed for this track, if there is to be no tendency to skid?

658. What is the minimum speed for the track of Problem 657, if $f=0.4$ and skidding is impending? *Ans. 31.6 mi per hr.*

659. Would it be possible for a motorcycle to travel in a horizontal plane around a track banked 90° with the horizontal?

660. A railway curve of 800-ft radius was built for equal rail pressures at a speed of 30 mi per hr. What is the flange pressure on a 100,000-lb car traveling 60 mi per hr?

661. If the center of gravity of the car in Problem 660 is 6 ft above the rails and $G = 4.9$ ft, what is the maximum allowable speed without tipping?

662. Why are railroad curves given a gradually increasing radius instead of simply making the straight track tangent to the arc of a circle?

663. A motorcycle travels horizontally around a track 100 ft in diameter. The upper portion of the track is banked 90° with the horizontal. If $f = 0.6$, what are the minimum speed of the motorcycle and its position relative to the track? Can this position be maintained?

664. A homogeneous cylinder 2 ft high and 6 in. in diameter rests on a revolving horizontal platform at a distance of 3 ft from the vertical axis of revolution (center to center). If $f = 0.3$ for the platform and cylinder, and the cylinder is to retain its original position, what is the maximum number of rpm the cylinder can make?

665. A block slides down a smooth plane 50 ft long and inclined 30° with the horizontal. The lower edge of the plane is 30 ft above the ground. Where will the block hit the ground? What is its striking velocity?

666. A ball starts from rest at A on the smooth curve in Fig. 468, and leaves the 30° plane at B and hits the ground at C. Compute the distances x and y.

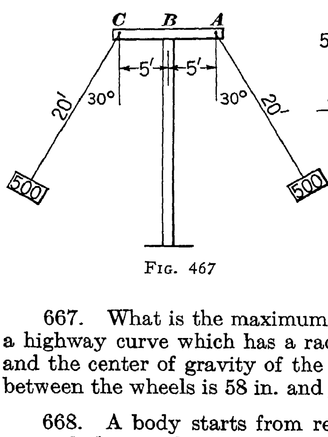

FIG. 467

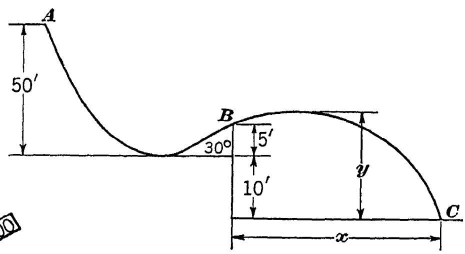

FIG. 468

667. What is the maximum safe speed for a 3,000-lb automobile around a highway curve which has a radius of 300 ft? The highway is banked 10° and the center of gravity of the car is 26 in. above the road. The distance between the wheels is 58 in. and $f = 0.4$.

668. A body starts from rest and slides down any smooth curve in a vertical plane or down a smooth inclined plane. Show that the time required for the body to move between any two elevations is a function of the slope of the curve.

669. If the radius of the loop in Fig. 465 is R, what is the minimum value which h can have in order that a car may pass the point A safely? *Ans. 2.5 R.*

670. A pilot pulls out of a power dive at a speed of 650 mi per hr. If his maximum acceleration is not to exceed $9g$, what is the minimum radius of the plane's path? What is the maximum pressure on the plane's seat, if the pilot weighs 175 lb?

671. A 40-lb weight slides down a circular path AB, Fig. 469, for which the frictional resistance is 10 lb and is always tangent to the surface of the curved path. The weight leaves the curved path at B and hits the ground at C. Determine: (a) the horizontal speed at B, (b) the striking speed at C, and (c) the distance x.

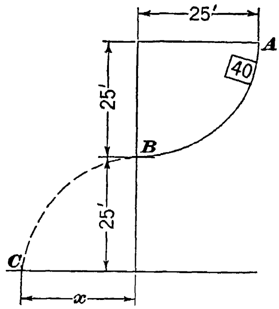

FIG. 469

CHAPTER 17

ROTATION[1]

151. Rotation of a Homogeneous Body Which Has a Plane of Symmetry Perpendicular to the Axis of Rotation.—Fortunately a large majority of the practical problems involving rotation deal with bodies which come under the above classification or the still more limited case where the axis of rotation is perpendicular to a plane of symmetry and passes through the center of gravity of the body.

Flywheels, motor and generator armatures, and turbine rotors are examples of rotating bodies which come under the latter classification, while plate cams, connecting-rods, and eccentrics are examples of the first type of rotating body.

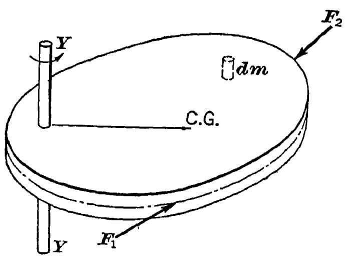

FIG. 470

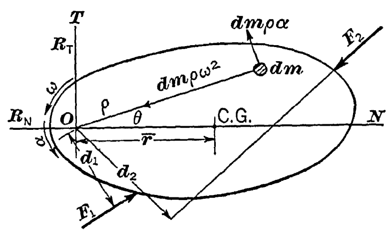

FIG. 471

Fig. 470 represents any homogeneous body, such as a plate cam, with a plane of symmetry perpendicular to the axis YY, about which it is rotating. The rotation is due to the unbalanced forces F_1 and F_2, which act in the plane of symmetry, and the reactions at the shaft.

Let it be assumed that the entire mass of the body is made up of an infinite number of elementary rods which are perpendicular to the plane of symmetry, as indicated by dm in Fig. 470. If each of these rods is compressed into a disk of differential thickness, without change in the cross-section or mass, then the mass of the entire rotating body may be considered as being concen-

[1] Some instructors may prefer to introduce Work and Energy before Rotation. Chapters 17 and 18 can be interchanged if desired.

trated in the plane of symmetry along with the forces F_1 and F_2 and the reactions at the shaft.

Fig. 471 represents the body of Fig. 470 after it has been reduced to a plate of differential thickness located at the plane of symmetry, with the mass of the elementary rod after it has been compressed represented by dm. Thus, by this process of reasoning, the entire system of forces which act on the body (except the weight which acts parallel to the axis of rotation and therefore has no effect on the rotation about the axis through O) has been reduced to a coplanar system which is perpendicular to the fixed axis through O about which the body is rotating.

Assume that the thin plate, Fig. 471, is turning about the axis through O, perpendicular to the plane of the paper, with an angular velocity ω and an angular acceleration α at any given instant, under the action of the unbalanced forces F_1 and F_2 and R_N and R_T.

The particle of mass dm will rotate about O in a circular path of radius ρ. According to Arts. 123 and 131, this particle of mass dm will receive two accelerations, $\rho\,\alpha$ in a tangential direction and $\rho\,\omega^2$ along the radius and directed toward the center of rotation O. There will, therefore, be two effective forces, $dm\,\rho\,\alpha$ and $dm\,\rho\,\omega^2$, acting on it. These forces are shown in Fig. 471.

Each and every other particle of mass in the body will also have two similar effective forces. Since all of these effective forces lie in the same plane, which is the plane of symmetry, their resultant must be either a force (when not rotating about an axis through the center of gravity, Art. 154) or a couple (when rotating about an axis through the center of gravity, Art. 152) in the plane of symmetry. The resultant of such a force system can be definitely determined by summing the components of the effective forces along any two rectangular axes in the plane and then locating the action line of the resultant by equating the moment of the resultant to the algebraic sum of the moments of the effective forces. However, for the purpose of solving problems it is generally more convenient to know the value of the resultant moment of the effective forces and the two rectangular components of the resultant effective force. These quantities will now be determined.

152. Resultant Moment of the Effective Forces.—According to the D'Alembert Principle, the resultant of the effective

forces for all particles of a given body is identical with the resultant of the external forces acting on the body. Therefore, by the principle of moments, the algebraic sum of the moments of all the effective forces for all the particles must be equal to the algebraic sum of the moments of the external forces. In Fig. 471, with moments being taken with respect to O,

$$F_1\, d_1 - F_2\, d_2 = \Sigma dm\ \rho\ \alpha\ \rho$$

$$F_1\, d_1 - F_2\, d_2 = \alpha \int dm\ \rho^2$$

Since $\int dm\ \rho^2 = I$,

$$F_1\, d_1 - F_2\, d_2 = I\ \alpha$$

The left-hand side of the foregoing equation is the resultant torque or external turning moment acting on the body. Since α, the angular acceleration, is always in the direction of the resultant torque, it is convenient to let $F_1 d_1$ represent the larger torque; otherwise, a negative quantity will be equated to a positive quantity. The equation then becomes

$$\text{Resultant Torque} = I\ \alpha$$

This equation bears the same relationship to rotation that the equation $\Sigma F = \frac{W}{g}\, a$ does to rectilinear translation and therefore may be employed in much the same manner.

In rectilinear translation $\frac{W}{g}\, a$ was shown to represent the inertia or resistance of a body to any change in its rectilinear motion. In a similar manner, in the equation Resultant Torque $= I\ \alpha$ for rotation, $I\ \alpha$ represents the inertia or resistance of a body to any change in its rate of rotation. Since the "Resultant Torque" is a couple, the term $I\ \alpha$ must also be a couple. If a couple equal to $I\ \alpha$ reversed is added to a body which receives an angular acceleration α because of an applied "Resultant Torque," that body will be in artificial equilibrium. The addition of the reversed $I\ \alpha$ couple extends the inertia method of solution used in rectilinear translation to problems of rotation.

EXAMPLE 1

A pulley 8 ft in diameter and weighing 2,000 lb is supported in bearings 6 in. in diameter. The tensions on the tight and slack sides of the belt are 500 and 350 lb, respectively. The radius

of gyration of the pulley is 3.5 ft, and f for the bearings is 0.08. How many rpm will the pulley make 20 sec after it starts from rest? How many feet of belt will pass over the pulley in 30 sec?

In Fig. 472 is the free-body diagram for the pulley.

$$\text{Frictional force } F=(2{,}000+350+500)\ 0.08$$
$$F=228 \text{ lb}$$

$$\text{Resultant torque}=(500-350)\ 4-228\times\frac{1}{4}=543 \text{ ft-lb}$$

$$543=\frac{2{,}000}{32.2}\times 3.5^2\,\alpha$$
$$\alpha=0.715 \text{ rad. per sec per sec}$$
$$\omega=\omega_0+\alpha\,t \text{ (Art. 132)}$$
$$\omega=0+0.715\times 20=14.2 \text{ rad. per sec}$$
$$\text{rpm}=\frac{14.2\times 60}{2\pi}=136.8$$
$$a=r\,\alpha=4\times 0.715=2.86 \text{ ft per sec per sec}$$
$$s=v_0\,t+\frac{1}{2}\,a\,t^2 \text{ (Art. 117)}$$
$$s=\frac{1}{2}\times 2.86\times 30^2=1{,}286 \text{ ft}$$

Inertia Method.—Let the reversed $I\,\alpha$ couple be represented, as in Fig. 473, opposite in sense to the angular acceleration α. Then equilibrium is established, and we can write

$$\Sigma M_O=0$$
$$(500-350)\ 4-228\times\frac{1}{4}-\frac{2{,}000}{32.2}\times 3.5^2\,\alpha=0$$
$$\alpha=0.715 \text{ rad. per sec per sec}$$

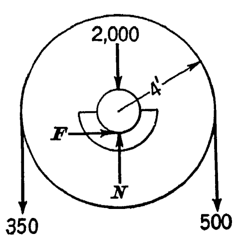

FIG. 472

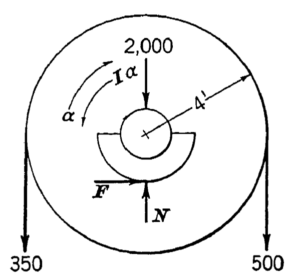

FIG. 473

EXAMPLE 2

A 500-lb cylinder 4 ft in diameter, Fig. 474 (*a*), has its axis of symmetry horizontal. The cylinder can turn freely about this axis. A 100-lb weight is supported by a cord which is wrapped about the cylinder. What are the angular acceleration of the cylinder, the tension in the cord, and the velocity of the weight after moving 20 ft, if its initial velocity was 10 ft per sec?

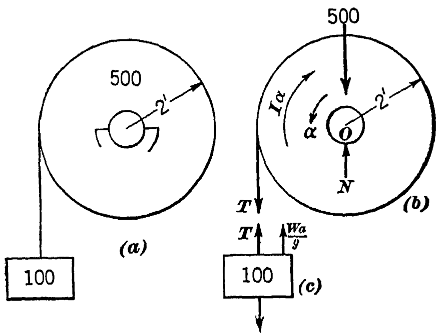

FIG. 474

This problem will be solved by the inertia method. Since the acceleration of the 100-lb weight is downward, its inertia force is upward. The angular acceleration of the drum is counterclockwise, and the inertia couple $I\alpha$ therefore is clockwise. In Fig. 474 (*b*),

$$\Sigma M_O = 0$$

$$2T - \frac{1}{2} \times \frac{500}{32.2} \times 2^2 \alpha = 0$$

$$T = 15.5\,\alpha \tag{1}$$

The second free body, Fig. 474 (*c*), has a motion of rectilinear translation.

$$T + \frac{100}{32.2}a - 100 = 0$$

Since $a = r\alpha = 2\alpha$,

$$T + 6.22\,\alpha - 100 = 0 \tag{2}$$

Equations (1) and (2) may now be solved for α and T.

$$\alpha = 4.6 \text{ rad. per sec per sec}$$
$$T = 71.5 \text{ lb}$$

Since $v^2 = v_0^2 + 2\,a\,s$ (Art. 117),

$$v^2 = 100 + 2 \times 9.2 \times 20$$
$$v = 21.6 \text{ ft per sec}$$

PROBLEMS

672. Fig. 475 represents a 400-lb drum with a brake applied to the outer surface. If a constant force of 200 lb is applied to the rope which is wound around the drum, and the drum is turning at a rate of 30 rpm when the brake goes into action, what force P must act on the brake to stop the drum in 10 sec? Assume that $k = 1.5$ ft and $f = 0.3$. *Ans. 347.9 lb.*

673. Determine the force P, Fig. 476, required to bring the 100-lb weight to rest after it has traveled 100 ft. The 500-lb drum is making 60 rpm when the brake is applied. Assume that $k = 2.5$ ft and $f = 0.5$.

674. A turbine rotor weighing 2,000 lb has the steam shut off when it is turning 1,800 rpm. If the rotor bearings are 6 in. in diameter, $f = 0.02$ for the bearings, $k = 3$ ft, and windage is neglected, how long will the rotor continue to turn?

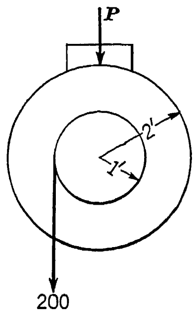

FIG. 475

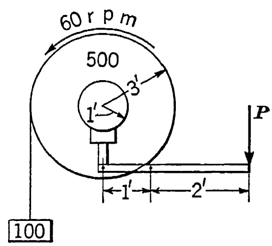

FIG. 476

675. In Problem 673, Fig. 476, let the 100-lb weight become a 100-lb force and let $P = 100$ lb. If the 500-lb drum is mounted on a 6-in. diameter shaft and $f = 0.08$ for the bearings, how many rpm are being made 20 sec after the brake begins to act?

676. Determine the distance moved by the 300-lb weight in Fig. 477 in 10 sec after starting from rest, if $f = 0.2$ for the plane and $k = 18$ in. for the drum.

677. A steel bar 2 in. in diameter and 8 ft long is free to rotate about a horizontal axis through O, Fig. 478. What is the angular acceleration of the rod when it passes through the 30° position indicated in Fig. 478? What are its accelerations when it has moved 30° and 45° further along in its swing?

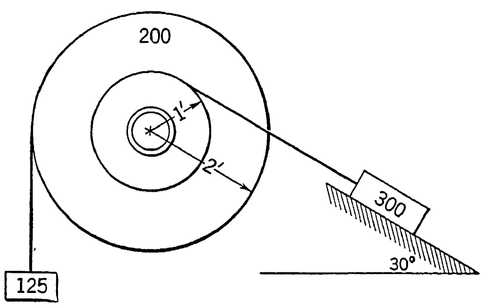

FIG. 477

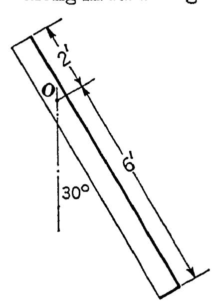

FIG. 478

153. Determination of the Normal and Tangential Components of the Resultant Effective Forces.—In Fig. 471 the N axis is drawn through the center of gravity and perpendicular to the axis of rotation through O, the point where the axis of rotation pierces the plane of symmetry. The distance between O and the center of gravity is represented by the quantity $\bar{r}$. The T axis is perpendicular to the N axis at O.

The algebraic sum of the components of all the effective forces parallel to the N axis is

$$\Sigma F_N = -\int dm\ \rho\ \omega^2 \cos\theta - \int dm\ \rho\ \alpha \sin\theta$$
$$\Sigma F_N = -\omega^2 \int dm\ \rho \cos\theta - \alpha \int dm\ \rho \sin\theta$$

$\int dm\ \rho \cos\theta$ is the algebraic sum of the moments of all the differential masses about the T axis. According to Art. 86,

$$\int dm\ \rho \cos\theta = M\ \bar{n} = M\ \bar{r}$$

In a similar manner, $\int dm\ \rho \sin\theta$ is the algebraic sum of the moments of all the differential masses about the N axis. Since the center of gravity is on the N axis, $\int dm\ \rho \sin\theta = 0$ and the component of the resultant effective force parallel to the N axis is

$$\Sigma F_N = -M\ \bar{r}\ \omega^2$$

The negative sign simply indicates that the force is directed toward the center of rotation O.

The algebraic sum of the components of the effective forces parallel to the T axis is

$$\Sigma F_T = \int dm\ \rho\ \alpha \cos\theta - \int dm\ \rho\ \omega^2 \sin\theta$$
$$\Sigma F_T = \alpha \int dm\ \rho \cos\theta - \omega^2 \int dm\ \rho \sin\theta$$

As just demonstrated, $\int dm\ \rho \cos\theta = M\ \bar{r}$ and $\int dm\ \rho \sin\theta = 0$. The component of the resultant effective force parallel to the T axis is then

$$\Sigma F_T = M\ \bar{r}\ \alpha$$

154. Location of the Lines of Action of the Normal and Tangential Components of the Resultant Effective Forces.—In Art. 153 the components of the resultant effective force parallel to the N and T axes were shown to be $M\ \bar{r}\ \omega^2$ and $M\ \bar{r}\ \alpha$.

Since the components of a resultant force can act at any point along the line of action of the resultant force, the positions of the

lines of action of $M\,\bar{r}\,\omega^2$ and $M\,\bar{r}\,\alpha$ can be determined in the following manner:

In Fig. 479 the N axis is taken as the line of action of the normal component $M\,\bar{r}\,\omega^2$. Then, if the point at which the tangential component $M\,\bar{r}\,\alpha$ intersects the normal component $M\,\bar{r}\,\omega^2$ can be located, a point on the resultant effective force will be determined. Therefore, the action lines of the two components $M\,\bar{r}\,\omega^2$ and $M\,\bar{r}\,\alpha$ are determined.

In Art. 152 the algebraic sum of the moments of the effective forces about the axis through O was shown to be $I\,\alpha$. By the principle of moments, the algebraic sum of the moments of the normal and tangential components of the resultant effective force about O must also be equal to $I\,\alpha$. In Fig. 479,

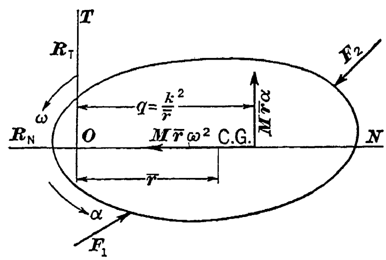

FIG. 479

$$\Sigma M_O = M\,\bar{r}\,\alpha\,q = I\,\alpha = M\,k^2\,\alpha$$
$$q = \frac{k^2}{\bar{r}}$$

Thus, the action lines of the normal and tangential components of the resultant effective force, for a body with a plane of symmetry normal to the axis of rotation, have been definitely located.

(*a*) The normal component of the resultant effective force is $M\,\bar{r}\,\omega^2$. It acts toward the axis of rotation along the perpendicular dropped from the center of gravity to the axis of rotation.

(*b*) The tangential component of the resultant effective force acts in the plane of symmetry perpendicular to the normal component at a distance $\frac{k^2}{\bar{r}}$ from the center of rotation. Its sense or direction is the same as that of α.

If the foregoing results are applied to a body, such as a flywheel or turbine rotor, which rotates about an axis normal to the plane of symmetry and passing through the center of gravity of the body, the quantity $\bar{r}$ becomes zero. The components $M\,\bar{r}\,\omega^2$ and $M\,\bar{r}\,\alpha$ are therefore zero for rotating bodies of this type. However, by Art. 152 the resultant moment of the effective forces is $I\,\alpha$. Therefore, the resultant of all the effective forces for a body which

rotates about an axis through its center of gravity and normal to the plane of symmetry is a couple with the value $I\,\alpha$.

155. Solution of Problems Involving Kinetic Reactions. Either the resultant effective force method or the reversed resultant effective force method, or inertia force method, may be used when a complete solution of rotational problems is required. However problems which involve kinetic reactions, Arts. 145 and 149, can generally be more easily analyzed if the reversed resultant effective force method, or inertia force method, is used.

If the normal and tangential components of the resultant effective force are reversed and added to the free body, equilibrium is established and any of the principles of statics may be employed in the solution of the free body.

Solution by the effective force method depends on the following reasoning. According to the D'Alembert Principle the resultant effective force is equal to the resultant of the external forces; therefore, a summation of the components of the external forces along any line must be equal to the component of the resultant effective force along the same line.

From the discussions of Arts. 152, 153, and 154 and the reasoning of the preceding paragraph, the three equations which follow are written. These equations are true only for bodies which rotate about an axis which is normal to a plane of symmetry.

$$\text{Resultant Torque} = I\,\alpha \tag{1}$$

$$\Sigma F_N = M\,\bar{r}\,\omega^2 \tag{2}$$

$$\Sigma F_T = M\,\bar{r}\,\alpha \tag{3}$$

where ΣF_N is a summation of the components of all the external forces along the N or normal axis, Fig. 479, and ΣF_T is a similar summation along the T axis.

EXAMPLE 1

In Fig. 480 there is shown a 500-lb cable and brake drum A, supported in horizontal bearings. A 300-lb weight B is suspended from a cable which is wrapped around the drum; and C is a brake shoe which presses against the brake drum with a normal pressure of 200 lb. The radius of gyration of the entire rotating mass is 2.5 ft, and $f = 0.4$ for the brake. Determine the tension in the cable, the acceleration of the 300-lb weight, the angular velocity

of the drum 10 sec after starting from rest, and also the horizontal and vertical components of the bearing reactions.

Fig. 480 (*b*) is the free-body diagram for the rotating drum. Since the axis of rotation passes through the center of gravity, $\bar{r}=0$ and the second members in equations (2) and (3) of Art. 155 become zero. Thus, as was stated in Art. 154, the resultant of all the effective forces for a body of this type when rotating about an axis through the center of gravity and perpendicular to the plane of symmetry is a couple with the value $I\,\alpha$ in the direction of α.

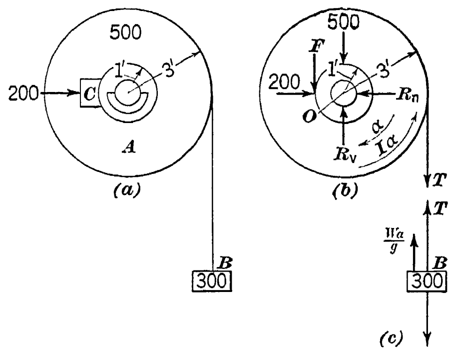

FIG. 480

Solution by the Inertia Method.—The tangential frictional force of the brake shoe is $200\times0.4=80$ lb. It is evident that the torque of the cable, or $T\times3$, is greater than 80×1; thus, the angular acceleration α of the drum is in a clockwise direction and the 300-lb weight accelerates downward.

Figs. 480 (*b*) and (*c*) are the free-body diagrams for the drum and the 300-lb weight, with the inertia couple $I\,\alpha$ and the inertia force $\frac{W}{g}\,a$ shown in their proper directions. In Fig. 480 (*b*),

$$\Sigma M_0=0$$

$$3T-80\times1-\frac{500}{32.2}\times2.5^2\,\alpha=0$$

$$3T-80-97.2\,\alpha=0 \tag{1}$$

In Fig. 480 (*c*),

$$\Sigma V=0$$

$$300-T-\frac{300}{32.2}\,a=0 \tag{2}$$

Since the linear acceleration of B is equal to the tangential acceleration of a point on the circumference of drum A, $a=r\,\alpha$ by Art. 131; thus, $\alpha=\frac{a}{3}$. Substituting this value in equation (1) and solving equations (1) and (2) for a and T gives:

$$T=173.4 \text{ lb and } a=13.61 \text{ ft per sec per sec}$$

$$\omega=\omega_0+\alpha\,t \text{ (Art. 132)}$$

$$\omega=0+\frac{13.61}{3}\times 10$$

$$\omega=45.3 \text{ rad. per sec}$$

In Fig. 480 (b),

$$\Sigma F_H=0$$
$$R_H=200$$

$$\Sigma F_V=0$$
$$R_V=500+173.4+80=753.4 \text{ lb}$$

It will be observed that the horizontal reaction is directly proportional to the normal brake pressure. The vertical reaction, however, is affected by the acceleration of the system which in turn is controlled by the brake.

EXAMPLE 2

A homogeneous slender rod 9 ft long and weighing 64.4 lb swings in a vertical plane about a pin A at one end of the rod. If the rod starts from rest from the vertical position shown in Fig. 481 (a) and turns through an angle of 120°, what are the normal and tangential components of the pin reaction when the rod passes through the 120° position as indicated in Fig. 481 (b)?

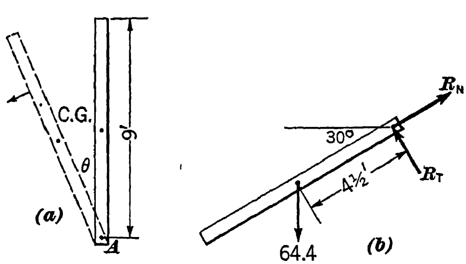

FIG. 481

Solution by the Effective Force Method.—Applying the Resultant 'orque equation (1), Art. 155, to the free body shown in Fig. 81 (*b*) gives

$$64.4\times 4.5\times 0.866=\frac{1}{3}\times\frac{64.4}{32.2}\times 9^2\,\alpha$$

For the 120° position,

$$\alpha=4.65 \text{ rad. per sec per sec}$$

$$M\,\bar{r}\,\alpha=\frac{64.4}{32.2}\times 4.5\times 4.65=41.8 \text{ lb}$$

For any position of the rod, α varies with the sine of the angle Fig. 481 (*a*), as is shown by the Resultant Torque equation.

$$64.4\times 4.5 \sin\theta=\frac{1}{3}\times\frac{64.4}{32.2}\times 9^2\,\alpha$$

$$\alpha=5.37 \sin\theta$$

$$\int\omega\, d\omega=\int\alpha\, d\theta \text{ (Art. 132)}$$

$$\int_0^{\omega}\omega\, d\omega=5.37\int_0^{120}\sin\theta\, d\theta$$

$$\frac{\omega^2}{2}=5.37\Big[-\cos\theta\Big]_0^{120}=8.05$$

$$\omega^2=16.1 \text{ rad. per sec at the } 120^\circ \text{ position*}$$

$$M\,\bar{r}\,\omega^2=\frac{64.4}{32.2}\times 4.5\times 16.1=144.9 \text{ lb}$$

Apply equation (2), Art. 155. In Art. 154 it was shown that $M\,\bar{r}\,\omega^2$ acts toward the center of rotation. For convenience this will be taken as the positive direction in the summation.

$$R_N-64.4\times 0.5=144.9$$

$$R_N=177 \text{ lb up to the right}$$

Apply equation (3), Art. 155. For the 120° position, α is in the counter-clockwise direction. The tangential component $M\,\bar{r}\,\alpha$ acts normal to the rod in a downward direction. This direction is taken as positive in the summation.

$$64.4\times 0.866-R_T=41.8$$

$$R_T=14 \text{ lb upward to the left}$$

* Some students prefer the Work and Energy Method for determining ω. In Art. 176 it is shown that Work $=\frac{1}{2}I\,\omega^2$. Thus,

$$64.4\,(4.5+4.5\times 0.5)=\tfrac{1}{2}\times 54\,\omega^2$$

$$\omega^2=16.1$$

Solution by the Inertia Method.—Fig. 482 (*a*) shows the rod as a free body with the reversed normal and tangential components of the resultant effective force, or the inertia forces, acting. This free body is in equilibrium.

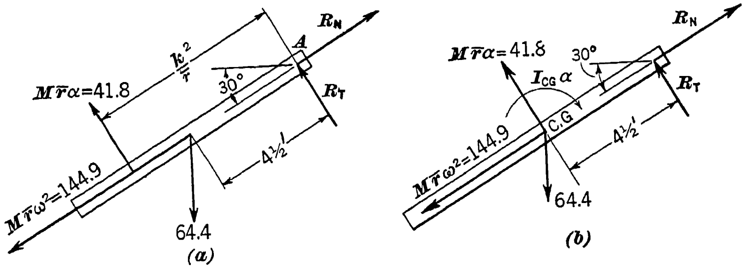

FIG. 482

$M\,\bar{r}\,\omega^2$ is obtained as in the previous solution.

$$\frac{k^2}{\bar{r}}=\frac{1}{3}\times\frac{64.4\times 9^2\times 32.2}{32.2\times 64.4\times 4.5}=6$$

$$\Sigma M_A=0$$
$$M\,\bar{r}\,\alpha\times 6-64.4\times 4.5\times 0.866=0$$
$$M\,\bar{r}\,\alpha=41.8$$

$$\Sigma F_N=0$$
$$R_N-144.9-64.4\times 0.5=0$$
$$R_N=177 \text{ lb up to the right}$$

$$\Sigma F_T=0$$
$$R_T+41.8-64.4\times 0.866=0$$
$$R_T=14 \text{ lb up to the left}$$

It is sometimes convenient to move the inertia forces to the center of gravity. This can be accomplished by adding to Fig. 482 (*a*) at the center of gravity pairs of equal, opposite forces $\left(\frac{W}{g}\,\bar{r}\,\omega^2 \text{ and } \frac{W}{g}\,\bar{r}\,\alpha\right)$ which are parallel to those shown. This system will be found to reduce to the system shown in Fig. 482 (*b*) because the couple

$$\frac{W}{g}\,\bar{r}\,\alpha\left(\frac{k_A^2}{\bar{r}}-\bar{r}\right)=\left(\frac{W}{g}\,k_A^2-\frac{W}{g}\,\bar{r}^2\right)\alpha=I_{CG}\,\alpha$$

By the transfer formula of Art. 109, $\frac{W}{g}\,k_A^2-\frac{W}{g}\,\bar{r}^2=I_{CG}$.

PROBLEMS

678. The 300-lb drum in Fig. 483 is turning 90 rpm. What force P must be applied to the brake arm to bring the 500-lb weight to rest after it moves 50 ft? Bearing friction is neglected, $f = 0.4$ for the brake shoe, and $k = 1.5$ ft for the drum. Determine the H and V components of the bearing reactions. *Ans.* *133 lb;* $R_H = 562.5$ *lb;* $R_V = 370.8$ *lb.*

679. When is the angular velocity of the rod in Example 2 greatest? What is its angular acceleration when $\theta = 180°$? When is the angular acceleration greatest?

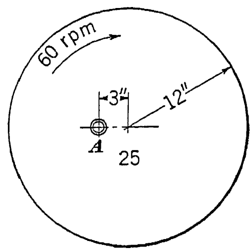

FIG. 485

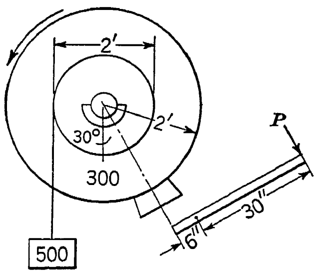

FIG. 483

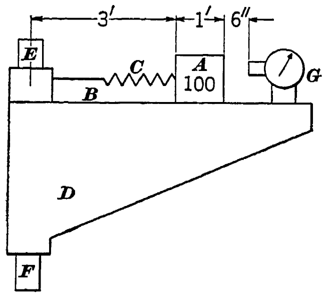

FIG. 486

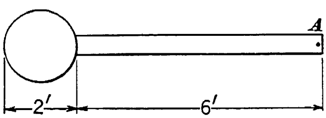

FIG. 484

680. If the rod in Example 2 has the pin moved in 1 ft from the end of the rod, what are the normal and tangential components of the pin reactions after the rod has swung through an angle of 210° from its starting position?

681. Fig. 484 represents a 50-lb cylinder, 2 ft in diameter, attached to a 10-lb rod which turns on a smooth pin at A. Determine the horizontal and vertical components of the pin reaction as the rod passes through a position 30° below that shown.

682. A homogeneous 25-lb disk is supported by a horizontal bearing at A, Fig. 485. It is turning 60 rpm when passing the position shown. What are the normal and tangential components of its bearing reactions after turning 60° clockwise from the position shown?

683. The 100-lb weight A in Fig. 486 is attached to a coil spring C and rests on the horizontal surface B, for which $f = 0.4$. When the bracket turns about the shaft EF at 45 rpm, the scales G show a reading of 100 lb. Determine the scale of the spring C.

156. Rotation of Bodies Acted Upon by Non-Coplanar Force Systems.—Generally, in the practical type of problem encountered

in the design of machinery, if the rotating body is not symmetrical with respect to a plane perpendicular to the axis of rotation, its form is such that it can be divided into parts which do have such planes of symmetry. The principles which have been developed for symmetrical bodies can then be applied to these parts.

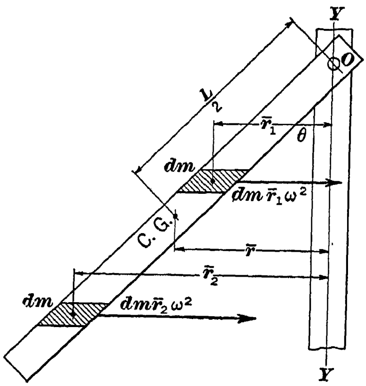

FIG. 487

The rods supporting the governor balls of a fly-ball governor are rotating bodies of this type.

Fig. 487 represents such a rod making an angle θ with the axis YY. If this rod is divided into small plates of differential thickness, each with a mass dm, each plate will have a plane of symmetry normal to the axis of rotation. The normal components of the effective forces for the differential plates will be $dm\,\bar{r}_1\,\omega^2$, $dm\,\bar{r}_2\,\omega^2$, $dm\,\bar{r}_3\,\omega^2$, etc. These forces are proportional to $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$, etc. The resultant of such a system of forces will be a force $M\,\bar{r}\,\omega^2$, perpendicular to the axis YY and passing through a point at a distance $\frac{2}{3}L$ from the point O. Here, L is the length of the rod and $\bar{r}$ is the perpendicular distance from the center of gravity of the rod to the axis YY. Also, $M\,\bar{r}\,\alpha$ is normal to $M\,\bar{r}\,\omega^2$ at a distance $\frac{k^2}{\bar{r}}$ from axis YY.

PROBLEMS

684. A 96.6-lb cone rotates about an axis which is 18 in. from the axis of symmetry. If the speed is changed from 30 rpm to 90 rpm in 10 sec, determine the normal and tangential components of its resultant effective force when the cone is making 60 rpm. Make a sketch and indicate the lines of action of the two components. *Ans. 177.5 lb; 2.82 lb.*

685. Fig. 488 represents a 100-lb homogeneous plate attached to the vertical shaft YY at points A and B and rotating at 60 rpm. Compare this case with that of Fig. 487.

686. In Fig. 489 a slender rod is attached to the vertical shaft YY by a pin at A. The rod weighs 40 lb and is 10 ft long. At what constant speed must the shaft turn if the rod is to remain at an angle of 60° with the shaft? Determine the horizontal and vertical components of the pin reaction at A.

687. A 20-lb sphere is attached to the end of a 10-lb rod 3 ft long. The rod is attached to the vertical shaft at A, Fig. 490. Determine the angle θ when the shaft is making 60 rpm.

688. If the rod in Fig. 487 has an angular acceleration, show that $M\bar{r}\alpha$ acts at a distance $\frac{2}{3}L \sin\theta$ from the axis YY.

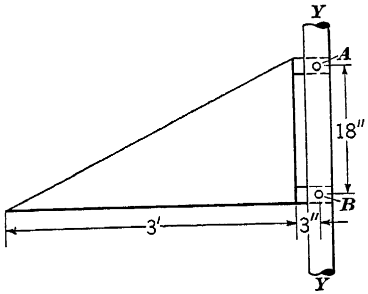

FIG. 488

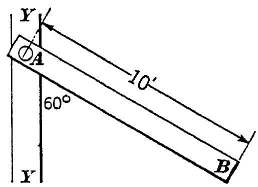

FIG. 489

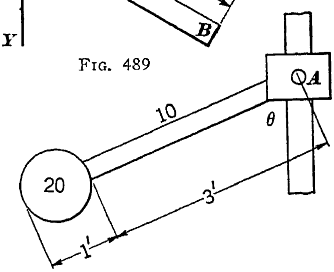

FIG. 490

157. **General Case of Rotation About Any Axis.**—As indicated in Art. 151, rotation of irregular or unsymmetrical objects has little application in engineering. The complete solution of this problem will be left to books on Theoretical Mechanics. It is advisable, however, for the engineer to have some understanding of the general problem of rotation.

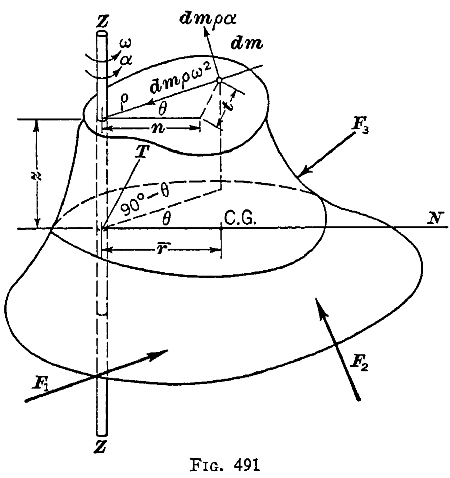

FIG. 491

Fig. 491 represents any irregular shaped body which is rotating about any axis ZZ with an angular velocity ω and an acceleration α, by reason of the action of the forces F_1, F_2, F_3, etc.

As in the discussions of Arts. 152, 153, and 154, the N axis is drawn through the center of gravity and perpendicular to the axis of rotation. The T axis is perpendicular to the N axis and the axis of rotation ZZ.

For convenience the top surface of the body, which contains the particle of mass dm, is taken perpendicular to the axis ZZ. The two effective forces for this particle of mass dm, which are $dm\ \rho\ \omega^2$ and $dm\ \rho\ \alpha$, are indicated on the figure.

A little study of this figure will soon show the student that it is impossible to condense the entire mass of the body into a thin plate of differential thickness and thus reduce the force system to a coplanar system, as was done in Art. 151, because of the unsymmetrical nature of the body. Each particle of mass in the body has two effective forces, similar to those shown in Fig. 491, acting on it. A summation of all these effective forces parallel to the N axis will give

$$\Sigma F_N = -\int dm\ \rho\ \omega^2 \cos\theta - \int dm\ \rho\ \alpha \sin\theta$$
$$\Sigma F_N = -M\ \bar{r}\ \omega^2, \text{ as in Art. 153}$$

A summation parallel to the T axis gives

$$\Sigma F_T = \int dm\ \rho\ \alpha \cos\theta - \int dm\ \rho\ \omega^2 \sin\theta$$
$$\Sigma F_T = M\ \bar{r}\ \alpha, \text{ as in Art. 153}$$

A summation parallel to the axis ZZ gives zero.

A summation of the moments of the effective forces with respect to the axis ZZ gives

$$\Sigma M_Z = \int dm\ \rho\ \alpha\ \rho$$

By Art. 152,

$$\Sigma M_Z = I\ \alpha$$

Examination of Fig. 491 shows that the effective forces also produce turning moments about the N and T axes. A summation of moments with respect to the N axis gives

$$\Sigma M_N = z\int dm\ \rho\ \alpha \cos\theta - z\int dm\ \rho\ \omega^2 \sin\theta$$
$$\Sigma M_N = \alpha\int z\ n\ dm - \omega^2\int z\ t\ dm$$

where the integral expressions are products of inertia which are usually rather difficult to determine.

In a similar manner a summation with respect to the T axis produces

$$\Sigma M_T = z \int dm\ \rho\ \alpha \sin\theta + z \int dm\ \rho\ \omega^2 \cos\theta$$
$$\Sigma M_T = \alpha \int z\ t\ dm + \omega^2 \int z\ n\ dm$$

It is thus seen that instead of three equations as in Art. 155 the unsymmetrical body gives six.

$$\Sigma F_N = -M\ \bar{r}\ \omega^2$$
$$\Sigma F_T = M\ \bar{r}\ \alpha$$
$$\Sigma F_Z = 0$$
$$R\ T_Z = I\ \alpha$$
$$R\ T_N = \alpha \int z\ n\ dm - \omega^2 \int z\ t\ dm$$
$$R\ T_T = \alpha \int z\ t\ dm + \omega^2 \int z\ n\ dm$$

where RT_Z, RT_N, and RT_T, are the resultant torques of the external forces F_1, F_2, F_3, etc., about the Z, N, and T axes. It is easily seen that the solution of these six equations can become a very difficult process.

158. Simple Circular Pendulum.—A simple circular pendulum consists of a small particle of mass attached to the free end of a weightless cord, which is free to swing in a vertical plane.

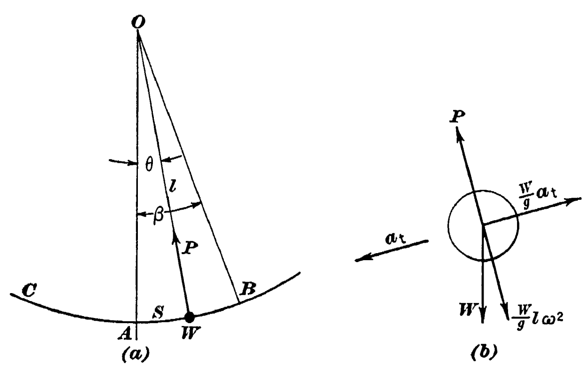

FIG. 492

In Fig. 492 (a), a small mass weighing W lb is attached to the cord l ft long, which is fixed at O. The weight moves back and forth from A to B to C to A along the arc.

The angular displacement θ, Fig. 492 (a), is taken as positive to the right. Fig. 492 (b) is the free-body diagram for the weight W, with the inertia forces shown.

$$\Sigma F=0 \text{ (tangentially)}$$

$$\frac{W}{g}a_t-W\sin\theta=0$$

$$a_t=g\sin\theta$$

When the amplitude of the pendulum (the maximum angular displacement θ to the right or left of OA) is small, $\sin\theta=\theta$.

$$a_t=g\,\theta$$

But,

$$\alpha=\frac{a_t}{l}=\frac{g\,\theta}{l}=\frac{d^2\theta}{dt^2}$$

where α is the angular acceleration of the pendulum.

Since α is opposite to the positive displacement θ,

$$\alpha=-g\frac{\theta}{l}$$

$$\frac{d^2\theta}{dt^2}=\frac{d}{dt}\left(\frac{d\theta}{dt}\right)=-\frac{g\,\theta}{l}$$

$$\int\left(\frac{d\theta}{dt}\right)\,d\left(\frac{d\theta}{dt}\right)=-g\int\frac{\theta\,d\theta}{l}$$

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2=-\frac{g}{l}\frac{\theta^2}{2}+C_1$$

Since the angular velocity $\frac{d\theta}{dt}=0$ when $\theta=\beta$, $C_1=\frac{g\,\beta^2}{2l}$.

$$\left(\frac{d\theta}{dt}\right)^2=\frac{g\,\beta^2}{l}-\frac{g\,\theta^2}{l}=\frac{g}{l}(\beta^2-\theta^2)$$

$$\frac{d\theta}{dt}=\sqrt{\frac{g}{l}}\,\sqrt{\beta^2-\theta^2}$$

$$\int_0^t dt=\sqrt{\frac{l}{g}}\int_0^\beta\frac{d\theta}{\sqrt{\beta^2-\theta^2}}$$

$$t=\sqrt{\frac{l}{g}}\sin^{-1}\frac{\beta}{\beta}=\sqrt{\frac{l}{g}}\frac{\pi}{2}$$

This is the time required for the pendulum to swing from A to B. The time to swing from A to B to C to A, or the period of the pendulum, is

$$T=4t=2\pi\sqrt{\frac{l}{g}}$$

This expression does not contain θ; therefore, the period of vibration is independent of the amplitude of the swing when the amplitude (angle β, Fig. 492) is less than about 4°.

Since the tangential component of the inertia force is perpendicular to the cord, the tension in the cord is not affected by this force. A summation along the cord gives

$$P-W\cos\theta-\frac{W}{g}l\,\omega^2=0$$

which will give a maximum value for P when the weight is at A, Fig. 492 (a).

PROBLEMS

689. What is the time required for a simple pendulum 100 ft long to swing from one extreme of its path to the other? *Ans. 5.54 sec.*

690. A simple pendulum 5 ft long makes 70 complete vibrations in 3 min. What is the value of the acceleration of gravity for the location of the pendulum?

691. If $g=32.2$, what length should a simple pendulum have in order that its period will be 1 sec?

159. Compound Pendulum.—A physical body which is free to swing or oscillate, because of the action of gravity, about any horizontal axis which does not pass through the center of gravity of the body, is a compound pendulum. The period or time of oscillation of such a pendulum may be determined in the following manner.

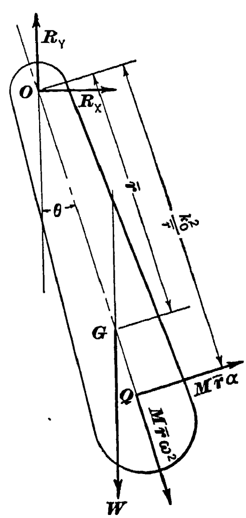

FIG. 493

In Fig. 493 the horizontal axis passes through point O; G is the center of gravity of the pendulum; and the reversed effective forces, or inertia forces, have been added to the free body.

$$\Sigma M_O=0$$

$$\frac{W}{g}\bar{r}\alpha\frac{k_0^2}{\bar{r}}-W\bar{r}\sin\theta=0$$

For small amplitudes, as in Art. 158,

$$\alpha = g \frac{\bar{r}}{k_0^2} \sin \theta$$

$$\alpha = g \frac{\bar{r}\,\theta}{k_0^2}$$

If the angular acceleration of the compound pendulum is equated to that of the simple pendulum, Art. 158,

$$\alpha = \frac{g\,\bar{r}\,\theta}{k_0^2} = \frac{g\,\theta}{l}$$

$$\frac{k_0^2}{\bar{r}} = l$$

Therefore, if the length l of a simple pendulum is made equal to the length $\frac{k_0^2}{\bar{r}}$ for a compound pendulum, the two pendulums will have the same angular accelerations and the same periods. By Art. 158, the period of the simple pendulum is

$$T = 2\pi \sqrt{\frac{l}{g}}$$

Then, for the compound pendulum,

$$T = 2\pi \sqrt{\frac{k_0^2}{g\,\bar{r}}}$$

This expression for the period of a compound pendulum is accurate only when the amplitude of the swing is such that sin θ can be taken equal to θ.

160. Center of Oscillation or Center of Percussion.—The point Q on the compound pendulum, Fig. 493, is the center of oscillation or center of percussion. It is the point at which the center of mass of the pendulum may be considered concentrated without change in the period of vibration.

The center of oscillation and the center of rotation may also be interchanged without change in the period of oscillation. The pendulum in Fig. 493 may thus be suspended from either the point O or the point Q and its period will be the same.

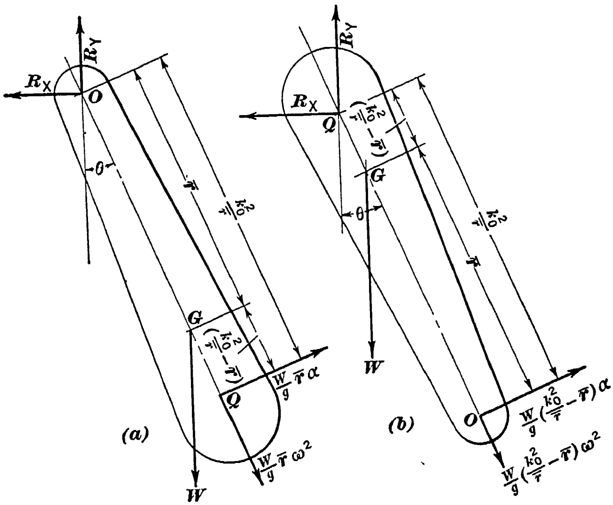

FIG. 494

In Fig. 494 (*a*) the pendulum is shown as a free body, with O the point of suspension; and, in Fig. 494 (*b*), Q is the point of suspension. In Fig. 494 (*a*),

$$\Sigma M_0 = 0$$

$$\frac{W}{g}\bar{r}\,\alpha\,\frac{k_0^2}{\bar{r}} - W\,\bar{r}\sin\theta = 0$$

$$\alpha = g\,\frac{\bar{r}\sin\theta}{k_0^2}$$

In Fig. 494 (*b*),

$$\Sigma M_Q = 0$$

$$\frac{W}{g}\left(\frac{k_0^2}{\bar{r}} - \bar{r}\right)\alpha\,\frac{k_0^2}{\bar{r}} - W\left(\frac{k_0^2}{\bar{r}} - \bar{r}\right)\sin\theta = 0$$

$$\alpha = \frac{g\,\bar{r}\sin\theta}{k_0^2}$$

Since the angular acceleration of the pendulum is the same whether it is suspended from point O or from point Q, its period will also be the same.

The moment of inertia of any object, which may be suspended from a horizontal axis through any point O on the object and

caused to oscillate as a compound pendulum, may be easily determined experimentally. From Art. 159,

$$k_0^2 = \frac{T^2}{4\pi^2} g \bar{r}$$

$$I = \frac{W}{g} k_0^2 = \frac{W T^2 \bar{r}}{4\pi^2}$$

The distance $\bar{r}$ is determined by experimental balancing of the object on a knife-edge, and T is the observed period of oscillation about the axis O.

PROBLEMS

692. Find the period and the center of oscillation for a slender rod that is 6 ft long and weighs 10 lb, if the rod rotates around a horizontal axis through one end. *Ans. 2.21 sec; 4 ft.*

693. If the rod in Problem 692 is hanging in a vertical position, at what point along its length can it be struck a blow without producing a horizontal reaction at the point of support?

694. A 20-lb sphere 6 in. in diameter is attached to one end of a 5-ft slender rod which weighs 5 lb. If the rod swings on a horizontal axis through the free end of the rod, what is the period of oscillation? Where is the center of percussion?

695. A locomotive connecting-rod weighs 600 lb. The rod is suspended from a horizontal knife-edge passed through one of the bearing openings. The center of gravity is 3.25 ft from the supporting knife-edge. If the rod oscillates 75 times in 3 min, what is its moment of inertia with respect to a horizontal axis through the center of gravity of the rod and parallel to the knife-edge?

161. Simple Harmonic Motion.—Simple harmonic motion is a rectilinear vibratory motion of a body in which the acceleration is always proportional to the displacement from the mid-point of the path and is directed toward the mid-point. This is expressed mathematically by the equation

$$a = \frac{d^2 s}{dt^2} = -k\, s = -\omega^2\, s$$

where s is the displacement from the mid-point of the path and k is a constant which will be shown to be equal to ω^2; that is, $k = \omega^2$. The negative sign indicates that the acceleration and the displacement are always opposite in direction.

A conical pendulum will be used to illustrate simple harmonic motion. In Fig. 495 the conical pendulum AP is rotating about

the vertical axis AO with a constant angular velocity ω. The small weight P travels in a horizontal circular path with a constant angular velocity ω.

If the motion of P is projected on a vertical plane through ACB, P will appear to move back and forth along the diameter BC. The motion of P observed in this manner will be simple harmonic motion.

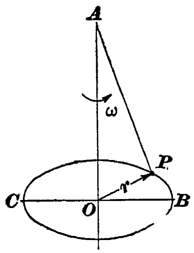

FIG. 495

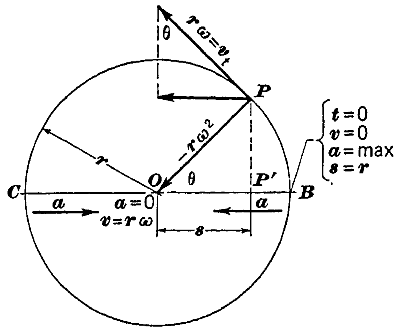

FIG. 496

Fig. 496 represents the horizontal plane $BPCO$ of Fig. 495. As P moves around its circular path with constant angular velocity ω, its projection P' on COB will move back and forth along COB. Since P is traveling on a circular path, it has an acceleration $r\,\omega^2$ along OP and a tangential velocity $v_t = r\,\omega$, Art. 131. Since P' is the projection of P, the velocity and acceleration of P' are equal to the components of the velocity and acceleration of P parallel to BC in the negative direction.

$$v_{P'} = -r\,\omega \sin\theta = -r\,\omega \sin\omega t$$

or

$$s = r\cos\theta = r\cos\omega t$$

$$v_{P'} = \frac{ds}{dt} = -r\,\omega\sin\omega t$$

where t represents the time required for P to travel from B to P along the curve, or $\theta = \omega t$.

$$a_{P'} = -r\,\omega^2\cos\theta = -r\,\omega^2\cos\omega t = -s\,\omega^2$$

or

$$a_{P'} = \frac{d^2s}{dt^2} = -r\,\omega^2\cos\omega t = -s\,\omega^2$$

Since P' requires the same time to go from B to C and back to B as P requires to travel around its circular path, the period for P' is

$$T=\frac{2\pi}{\omega}$$

and the frequency is

$$f=\frac{\omega}{2\pi}$$

The amplitude does not appear in these equations. Therefore, the period of vibration T and the frequency f, in vibrations per second, are independent of the amplitude.

The time required for P' to travel from B to any position P' is the same as the time required for P to travel from B to P. Thus,

$$t_{BP'}=\frac{\theta}{\omega}$$

EXAMPLE 1

If the radius r in Fig. 496 is 3 ft and the pendulum is making 120 rpm, what is the period of vibration for P'? What are the velocity and acceleration of P' when the angle θ is 30°? How long will it take P' to reach point O?

$$\omega=\frac{120\times 2\pi}{60}=4\pi \text{ rad. per sec}$$

$$T=\frac{2\pi}{4\pi}=0.5 \text{ sec}$$

$$v_{P'}=r\,\omega \sin\theta=3\times 4\pi\times 0.5=6\pi \text{ ft per sec}$$

$$a_{P'}=r\,\omega^2 \cos\theta=3\times(4\pi)^2\times 0.866=41.6\pi^2 \text{ ft per sec}$$

$$60°=\frac{2\pi}{6}\text{ rad.}$$

$$t_{P'O}=\frac{2\pi}{6\times 4\pi}=0.0833 \text{ sec}$$

EXAMPLE 2

A 100-lb weight is suspended from a coil spring. If the weight makes 60 complete vibrations per minute when set in motion, what is the scale of the spring in pounds per inch?

The reversed resultant effective force method, or inertia force method, will be used in solving this problem. When the 100-lb weight is set in motion, it moves in a manner similar to the point

P' in Fig. 496. The path of the weight becomes the diameter of an imaginary circle around which an imaginary point P is traveling at a constant angular velocity ω.

$$\omega=\frac{60\times 2\pi}{60}=2\pi \text{ rad. per sec}$$

Fig. 497 (a) is the free-body diagram for the weight when it is at rest due to the action of the spring A and the pull of gravity.

In Fig. 497 (b) the weight has been displaced any convenient distance, such as s inches, from the static position shown in Fig. 497 (a). Fig. 497 (b) is the free-body diagram for the weight at the instant at which it is released after having been given a displacement of s inches. The spring now exerts an upward pull of $100+s\,C$, where C is the scale of the spring in pounds per inch or the force required to elongate the spring 1 in. The weight is being accelerated upward; therefore, the reversed resultant effective force, or inertia force, acts downward. According to Art. 161, $a=s\,\omega^2$.

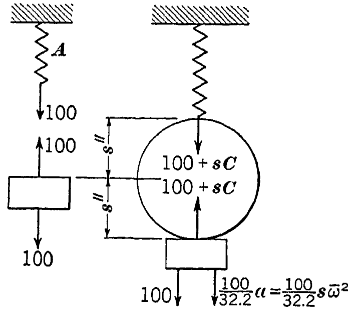

FIG. 497

Summing forces in the vertical direction gives

$$100+s\,C=\frac{100}{32.2}\times\frac{s}{12}(2\pi)^2+100$$
$$C=10.2 \text{ lb per in.}$$

PROBLEMS

696. A point moving with simple harmonic motion has an amplitude of 1 ft and a period of 1 sec. Determine the displacement, the velocity, and the acceleration of the point 0.3 sec after it leaves the end of its path. *Ans. 1.31 ft; 5.97 ft per sec; 12.18 ft per sec².*

697. In Fig. 498 is shown the crankpin circle of a simple steam engine. For the position P of the crankpin, determine the speed and acceleration of

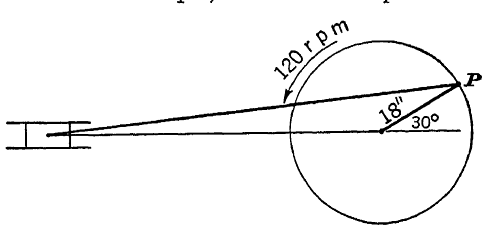

FIG. 498

the crosshead of the engine. The connecting-rod is assumed to be sufficiently long to allow the motion of the crosshead to be considered simple harmonic motion.

698. A 300-lb weight suspended from a coil spring elongates the coil spring 5 in. If this weight is set in vibration, how many vibrations per minute will it make? When the amplitude of its vibration is 4 in., what are its maximum velocity and maximum acceleration?

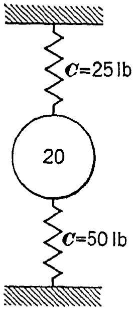

FIG. 499

699. An unknown mass is suspended from the spring of Problem 698. The mass vibrates 180 times per minute when it is set in motion. How much does the mass weigh? *Ans. 65.3 lb.*

700. A point moves in a straight line with an acceleration $a = -12\,s$. If its maximum velocity is 4 ft per sec, what are its period of vibration and its amplitude? *Ans. 1.815 sec; 1.155 ft.*

701. The 20-lb ball in Fig. 499 is attached to both springs. If the ball is displaced 4 in. from the center position and then released, with what velocity will it pass the center position? What is its period of vibration?

162. Why Rotating Bodies Need Balancing.—The necessity for balancing rotating bodies can be shown most easily by considering one or two examples.

In Fig. 500 (*a*), *A* is a vertical shaft projecting from a bearing *B* and passing through the center of gravity of the homogeneous wheel *C*. When the shaft and wheel rotate, there will be no side thrust on bearing *B* and, therefore, no need for balancing weights.

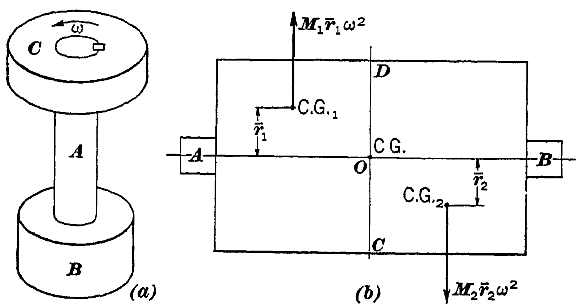

FIG. 500

If a hole is bored in the wheel so that the center of gravity of the wheel is no longer at the center of the shaft, then when the wheel rotates the end of the shaft will tend to wabble and a side thrust with variable direction will be developed at the bearing, causing wear.

This tendency to wabble is caused by the normal component $M\,\bar{r}\,\omega^2$ of the reversed resultant effective force, or inertia force, which acts horizontally through the center of gravity of the wheel and the center of the shaft. As the wheel turns, the force $M\,\bar{r}\,\omega^2$ also turns, causing the end of the shaft to wabble and producing a variable bearing pressure. The remedy for this condition of unbalance is self-evident. Add a balancing weight that will bring the center of gravity of the rotating mass back to the center of the shaft.

Consider next the motor rotor, Fig. 500 (*b*), supported in horizontal bearings *A* and *B* with the center of gravity of the entire rotor at *O* on the axis of rotation.

Let the rotor be divided into two halves by the plane *CD*. The centers of gravity of the two halves are at $C.G._1$ and $C.G._2$. This rotor is in static balance. If the rotor is turned in its bearings to any position, it will remain in that position. If, however, the rotor is caused to rotate, the bearings will be subject to excessive wear because, as indicated in Fig. 500 (*b*), the normal components of the reversed resultant effective or inertia forces, $M_1\,\bar{r}_1\,\omega^2$ and $M_2\,\bar{r}_2\,\omega^2$, form a couple which must be resisted by another couple at the bearings.

This condition of unbalance (*dynamic unbalance*) may be removed by adding weights to the lower left and upper right sections of the rotor in such a manner that the center of gravity of each half of the rotor will be brought back to the axis of rotation.

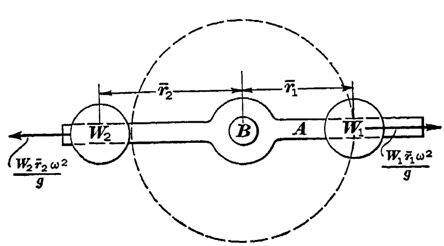

FIG. 501

163. Balancing in a Single Plane.—In Fig. 501, *A* is any symmetrical bar turning about a bearing *B* through its center of gravity; and W_1 is any weight attached to the bar at a distance $\bar{r}_1$ from the center of the bearing. When the bar rotates at a constant angular velocity ω, the reversed resultant effective force or an

unbalanced inertia force, $\frac{W_1 \bar{r}_1 \omega^2}{g}$ will act on W_1. This force can be balanced by placing another weight W_2 diametrically opposite W_1 in such a manner that $\frac{W_1 \bar{r}_1 \omega^2}{g} = \frac{W_2 \bar{r}_2 \omega^2}{g}$.

Since g and ω are constants, the condition for running balance is:

$$W_1 \bar{r}_1 = W_2 \bar{r}_2$$

This is also the condition for static balance.

If several weights are rotating in the same plane, these weights can be balanced by a single additional weight, also rotating in the plane of the given weights. The balancing of such a system of weights will now be illustrated by the solution of an example.

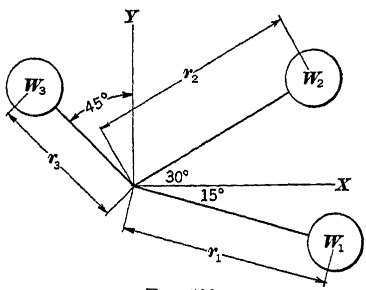

FIG. 502

EXAMPLE

Balance the three weights shown in Fig. 502 by a single weight rotating in the same plane at a radius of 15 in.

$W_1 = 10$ lb	$r_1 = 12$ in.
$W_2 = 15$ lb	$r_2 = 15$ in.
$W_3 = 20$ lb	$r_3 = 10$ in.

Each of the rotating weights will have a reversed resultant effective force, or an unbalanced radial inertia force, such as $\frac{W_1 r_1 \omega^2}{g}$, acting away from the center of rotation. Since g and ω^2 are common to each of these forces, the forces are proportional to $W_1 r_1$, $W_2 r_2$, and $W_3 r_3$. These unbalanced forces form a coplanar concurrent system. The resultant of the system is given by

$$R = W r = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

	X Component +	X Component −	Y Component +	Y Component −
$W_1 r_1 = 120$	116			31.1
$W_2 r_2 = 225$	195		112.5	
$W_3 r_3 = 200$		141.4	141.4	
	311	−141.4	253.9	−31 1

$\Sigma F_x = 169.6$ $\Sigma F_y = 222.8$

$$W\,r = \sqrt{169.6^2 + 222.8^2}$$
$$W\,r = 280$$
$$\tan\theta = \frac{222.8}{169.6} = 1.313;\ \theta = 52.7° \text{ with } X \text{ axis}$$

Since the resultant of the system is $W\,r = 280$ acting at 52.7° with the X axis, the system will be completely balanced by a weight of $\frac{280}{15} = 18.6$ lb acting at a radius of 15 in. and making an angle of 232.7° with the X axis.

PROBLEMS

702. What weight acting at a distance of 8 in. from the center of rotation will balance a 50-lb weight acting at a radius of 2.5 ft? *Ans. 187.5 lb.*

703. In Fig. 502, interchange the 30° and 45° angles, and determine the position at which a 15-lb weight must act to balance the system.

704. What weight placed half way between the 10-lb ball and the axis will balance the system shown in Fig. 503?

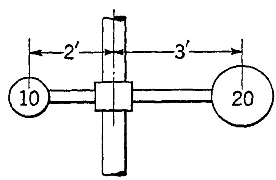

FIG. 503

705. A 40-lb sphere 6 in. in diameter and a 65-lb sphere 8 in. in diameter are connected by a rod of rectangular cross-section which is 4 ft long and weighs 24 lb. Determine the location of the center of the hole to be drilled in the rod for a vertical shaft if the assembly is to be dynamically balanced when mounted horizontally on the shaft.

164. Balancing of Shafts and Other Rotating Bodies.—Fig. 504 (a) represents a shaft turning in bearings at A and B with the weight W_1 eccentric a distance $\bar{r}_1$. This shaft will be *statically balanced* by any weight W_2 which is placed so that $W_1\,\bar{r}_1 = W_2\,\bar{r}_2$. However, when this shaft rotates, W_1 will develop an inertia or resultant reversed effective force $\frac{W_1\,\bar{r}_1\,\omega^2}{g}$ and W_2 will develop an equal inertia force $\frac{W_2\,\bar{r}_2\,\omega^2}{g}$, which will act as indicated in Fig. 504 (a). These forces form a couple $\frac{W_1\,\bar{r}_1\,\omega^2\,a}{g}$, which will induce an equal and opposite resisting couple $R_A\,b$ at the bearings. If the weight W_2 is placed diametrically opposite W_1, no couple will be produced and the system will be completely balanced statically and dynamically. It is thus seen that, if a single weight is to be *dynamically balanced* by another *single* weight, the balancing weight must be placed so that its center of gravity and that of the

weight to be balanced rotate in the *same plane* normal to the axis of rotation.

If the above condition is impossible, the single weight W_1 may be statically and dynamically balanced by two weights W_2 and W_3, Fig. 504 (*b*), each rotating in a plane normal to the axis of rotation for W_1.

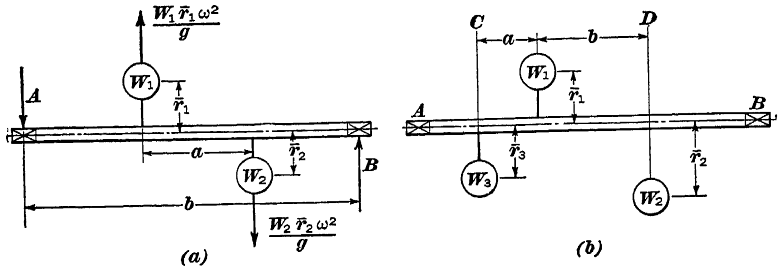

FIG. 504

The condition required for perfect dynamic balance is that no dynamic reaction shall be induced at either bearing A or bearing B. This will be accomplished when $\Sigma M_C = 0$ and $\Sigma M_D = 0$. Then,

$$W_1 \bar{r}_1 a - W_2 \bar{r}_2 (a+b) = 0$$
$$W_1 \bar{r}_1 b - W_3 \bar{r}_3 (a+b) = 0$$

In any given case all but two of the quantities in these equations will be known or can be assumed. It will then be possible to solve for the remaining two unknowns. These equations imply that the dynamic reactions at each of the bearings are zero.

The student sometimes has the idea that balancing reduces the entire bearing reaction to zero; but this is not true. The *only function of balancing is to reduce the dynamic components of the reactions to zero.* The static components cannot be balanced out.

EXAMPLE

In Fig. 504 (*b*), let $\bar{r}_1 = 6$ in.; $\bar{r}_2 = 10$ in.; $\bar{r}_3 = 12$ in.; $W_1 = 100$ lb; $a = 2$ ft; and $b = 3$ ft. Determine W_2 and W_3 for dynamic balance.

$$\Sigma M_D = 0$$
$$100 \times 6 \times 3 - W_3 \times 12 \times 5 = 0$$
$$W_3 = 30 \text{ lb}$$
$$\Sigma M_C = 0$$
$$100 \times 6 \times 2 - W_2 \times 10 \times 5 = 0$$
$$W_2 = 24 \text{ lb}$$

PROBLEMS

706. A horizontal shaft 6 ft long between bearings has a 180-lb disk keyed to it at a distance of 2 ft from the right bearing. The center of gravity of the disk is eccentric 4 in. Determine the balancing weights, if these weights are placed in planes 1 ft from the bearings and are eccentric 1 ft.

707. The shaft in Fig. 505 is to be balanced by placing weights in planes *AA* and *BB*. Compute these weights if they act at a 12-in. radius. *Ans.* $W_A = 0.5$ *lb;* $W_B = 7$ *lb.*

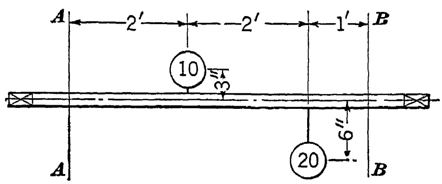

FIG. 505

708. Locate the exact positions of the balancing weights in Problem 707.

709. Fig. 506 shows a shaft with 50-lb and 75-lb weights eccentric 6 and 3 in. and located as indicated. Balance the shaft by placing a single weight in plane *A* and another in plane *B*, both balancing weights to be eccentric 10 in.

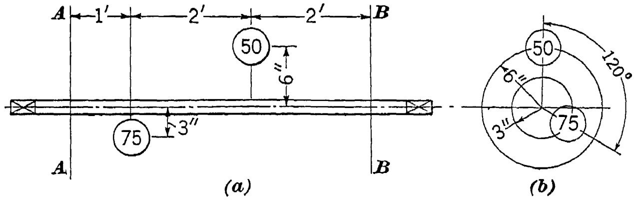

FIG. 506

165. Torsional Pendulum.—A body, Fig. 507 (*a*), supported by an elastic wire or slender rod fixed at one end is called a torsional pendulum if the body oscillates when it is given an angular displacement and released. The center of gravity of the mass M of the body is on the axis of the supporting wire. If the angular displacement θ is of such magnitude that the elastic limit of the wire or rod is not exceeded, the resisting or restoring torque which the wire exerts on the mass M is proportional to the angular displacement θ of the mass and is always in the opposite direction. Since Torque $= I\,\alpha$ (Art. 152), the angular acceleration α is also proportional to the torque and the displacement. Thus,

$$\alpha = -k\,\theta$$

where k is a constant and the negative sign indicates that the angular acceleration and the displacement are in opposite directions.

From the theory of strength of materials, the resisting or restoring torque which the wire exerts on the mass M is known to be

$$\text{Torque} = \frac{G\,\theta\,J}{L} \text{ in.-lb}$$

Here, G is the shearing modulus of elasticity, J is the polar moment of inertia of the cross-section of the wire with respect to the axis of the wire, and L is the length of the wire. Inches, pounds per square inch, inches^4, and inch-pounds are the units used.

If θ is positive in Fig. 507 (a), α is negative.

$$\frac{G\,\theta\,J}{12\,L} = I\,(-\alpha)$$

where I is the moment of inertia of the rotating mass with respect to the axis of the rod.

$$\alpha = -\frac{G\,J}{I\,L}\,\frac{\theta}{12}$$

Since $\theta = \frac{s}{r}$ and $\alpha = \frac{a}{r}$,

$$a = -\frac{G\,J}{I\,L}\,\frac{s}{12}$$

This equation is in the form $a = -\omega^2\,s$, as in simple harmonic motion, Art. 161. Therefore, $\omega^2 = \frac{G\,J}{12\,I\,L}$; and, since the period $T = \frac{2\pi}{\omega}$ for the torsional pendulum,

$$T = \frac{2\pi}{\sqrt{\frac{G\,J}{12\,I\,L}}} = 2\pi\sqrt{\frac{12\,I\,L}{G\,J}} = k\,\sqrt{I}$$

For any given wire, within the elastic limit of the material, $k = 2\pi\sqrt{\frac{12\,L}{G\,J}}$ is constant. Therefore, the period of the torsional pendulum is proportional to the square root of the moment of inertia of the oscillating mass M.

The torsional pendulum offers a convenient method for determining the moment of inertia of any irregular shaped mass, such as M_1 in Fig. 507 (*b*), if the mass M_1 is placed on the torsional pendulum in Fig. 507 (*a*), as indicated in Fig. 507 (*b*), with the C.G. of M_1 on the axis of the wire.

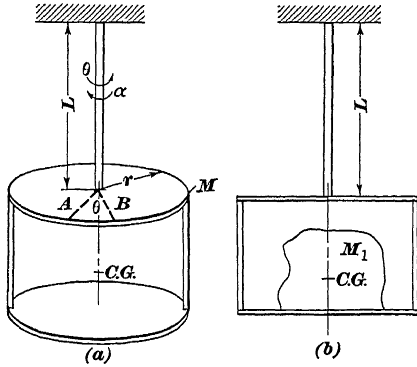

FIG. 507

For Fig. 507 (*a*), the constant $k=2\pi\sqrt{\frac{12\,L}{G\,J}}$ can be computed and the period T can be determined by observation. The period is proportional to $\sqrt{I}$. Hence, if T_1 is the period and I_1 is the moment of inertia of M_1 in Fig. 507 (*b*),

$$T_1 = T\sqrt{\frac{I+I_1}{I}}$$

PROBLEMS

710. A cast-iron cube 1 ft on each edge is attached to a vertical steel rod $\frac{1}{4}$ in. in diameter and 7 ft long. The upper end of the rod is rigidly fixed. Compute the period of oscillation and the torque required to give the weight an angular displacement of 60°. Assume that $G=12{,}000{,}000$ lb per sq in.; and cast iron weighs 450 lb per cu ft.

711. A solid circular disk, weighing 64.4 lb and 3 ft in diameter, is supported by a wire $\frac{1}{8}$ in. in diameter and 3 ft long and fixed at the upper end. The disk oscillates 5.2 times per min. If, when a gear is placed on the disk with its C.G. at the axis of the wire, the gear and the disk oscillate 3 times per minute, what is the moment of inertia of the gear with respect to the axis through the C.G.?

166. The Loaded Conical Pendulum Governor.—In Art. 148 it was demonstrated that the height h of a simple conical pendulum is inversely proportional to the square of the speed of rotation

and is not influenced by the length of the arm or the weight of the rotating mass. The governing action of the pendulum type governor is accomplished by the change in height h caused by variation in speed, which is transmitted to the steam valve of the engine through a linkage. At low speeds a small change of speed causes a relatively large change in h, while at high speeds a large change in speed is required to produce a small change in h. The sensitiveness of the pendulum governor at high speeds can be increased by loading it with a weight W_1, as in Fig. 508 (a).

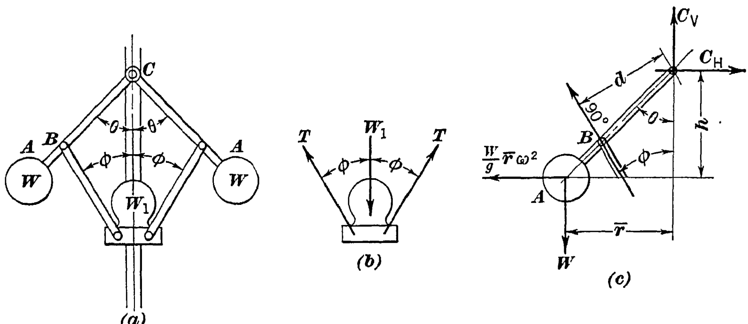

FIG 508

With the weight W_1 as the first free body, Fig. 508 (b), and a constant angular velocity ω,

$$\Sigma V=0$$

$$T=\frac{W_1}{2\cos\phi}$$

The second free body, Fig. 508 (c), is one of the weights W and its supporting rods. It generally is sufficiently accurate to neglect the weights of the rods, which are small in comparison with W.

$$\Sigma M_B=0$$

$$T\,d+W\,\bar{r}-\frac{W}{g}\,\bar{r}\,\omega^2\,h=0$$

PROBLEMS

712. In Fig. 508 (a), $AB=6$ in., $BC=12$ in., $W=30$ lb, $W_1=100$ lb, and $\theta=\phi=45°$. When the weight W_1 is at its lowest position, at what speed, in rpm, will the governor begin to function if the rods are weightless?

713. What load W_1 will be required in Problem 712, if the governor is to begin to function at 110 rpm?

REVIEW PROBLEMS

714. A force of 200 lb is applied to a cable which is wrapped around a 1,000-lb cylinder 2 ft in diameter. The cylinder is supported by resting in two 60° V-shaped bearings. If $f = 0.07$ for the surface of the cylinder in contact with the V bearings, what is the angular acceleration of the cylinder, and how much rope will unwind in 15 sec? *Ans. 2.065 rad. per sec; 232.5 ft.*

715. In Fig. 509, A is a solid cylinder 4 ft in diameter and weighing 10,000 lb. It turns in bearings 10 in. in diameter, for which $f = 0.015$. Determine the tension in the rope and the velocity of the 100-lb weight 10 sec after starting from rest.

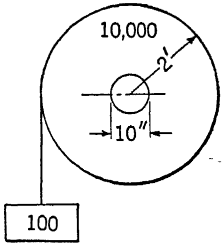

FIG. 509

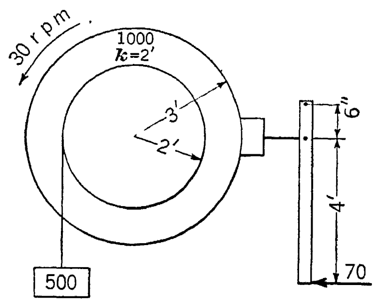

FIG. 510

716. The drum in Fig. 510 is turning 30 rpm when the brake is applied. If f for the brake is 0.5, what is the speed of the drum, in rpm, 10 sec after the brake is applied?

717. Determine the tension in each rope, Fig. 511, and the time required for the 300-lb weight to move 75 ft.

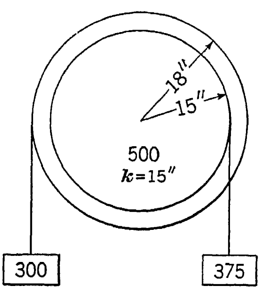

FIG. 511

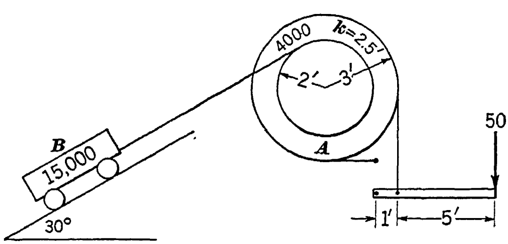

FIG. 512

718. In Fig. 512, A is an electrically driven mine hoist. If the car B is moving up the incline at 30 mi per hr, the force applied to the brake lever is 50 lb, $f = 0.4$ for the brake, and the car resistance is 200 lb per ton, how far from the surface must the power be shut off? *Ans. 55.4 ft.*

719. A 5,000-lb elevator is raised by having its rope wound on a 1,000-lb drum 3 ft in diameter. Determine the torque in foot-pounds which must be applied to the drum, if the elevator is to have an acceleration of 2 ft per sec per sec and $k = 2$ ft.

720. A cast-iron flywheel has an outside diameter of 7 ft and an inside diameter of 6 ft, and is 18 in. wide across the face. The shaft bearings are 6 in. in diameter and $f = 0.012$ for the bearings. How many revolutions will the wheel make in coming to rest from a speed of 120 rpm, if the mass of the hub and spokes is neglected?

721. Power is shut off when the 2,000-lb weight in Fig. 513 has an upward velocity of 60 ft per sec. If $f = 0.025$ for the shaft bearings, how long will the weight continue to rise?

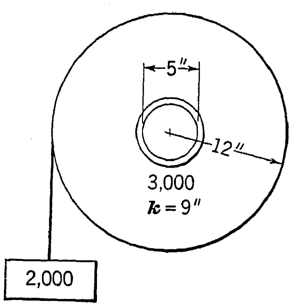

Fig. 513

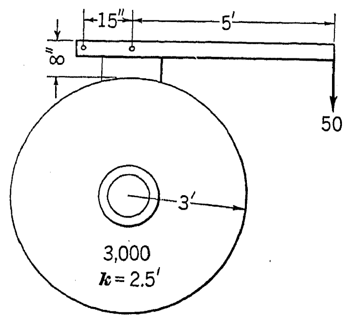

Fig. 514

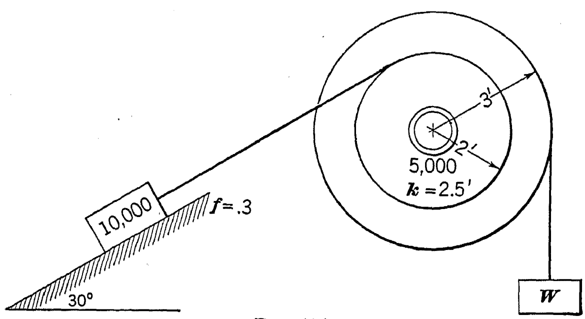

Fig. 515

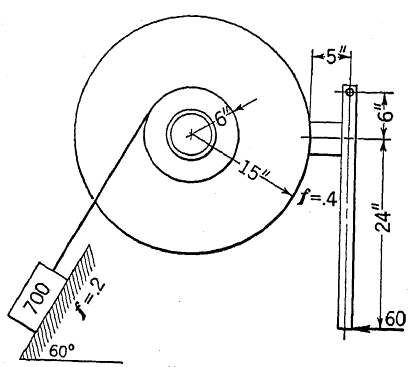

Fig. 516

722. The 3,000-lb drum, Fig. 514, is turning 300 rpm clockwise when the brake is applied. How long will it continue to turn, if $f = 0.4$ for the brake?

723. Compute the weight W, Fig. 515, if the 10,000-lb weight starts from rest and moves 75 ft down the plane in 10 sec.

724. Determine the velocity of the 700-lb weight, Fig. 516, at an instant 20 sec after it has attained a velocity of 10 ft per sec down the 60° plane. The rotating drum weighs 96.6 lb and its $k = 12$ in.

725. If the rod in Problem 686 and Fig. 489 is rigidly attached to the shaft YY and the shaft has a speed of 60 rpm, determine the bending moment at point A.

726. Fig. 517 represents a 60° steel sector 3 in. thick. The sector is free to turn about the horizontal axis through A. Determine the normal and tangential components of the bearing reactions when the sector is passing through a position 150° from the starting position. Steel weighs 490 lb per cu ft.

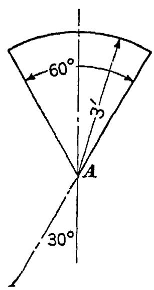

FIG. 517

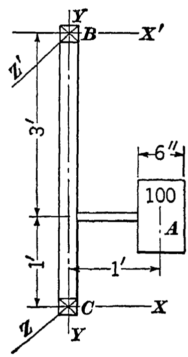

FIG. 518

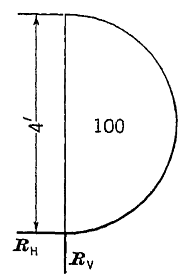

FIG. 519

727. A 20-lb sphere 1 ft in diameter is attached to the end of a 10-lb rod 6 ft long. The rod turns on a horizontal axis through a point 1 ft from the free end of the rod. In the starting position the sphere is directly above the bearing. Determine the H and V components of the bearing reactions when the sphere and rod have moved through an angle of 105°.

728. In Fig. 518, A is a 100-lb cylinder 6 in. in diameter, which is attached to a shaft by a slender weightless rod. If the shaft is turning at a constant speed of 30 rpm, what are the X, Y, and Z components of the bearing reactions at B and C for the position of the cylinder shown? *Ans.* $B_x = 32.6$ *lb;* $C_x = 2.05$ *lb;* $C_y = 100$ *lb;* $B_z = C_z = 0$.

729. Compute R_H and R_V after the semi-circular 100-lb plate in Fig. 519 has turned through 180°. The pin at R is frictionless.

730. Locate the center of percussion of the plate in Problem 729.

731. After the plate in Problem 729 has come to rest it is given a small angular displacement. What is the frequency of oscillation?

732. A 10-lb rod 8 ft long is supported by a horizontal axis through a point 1 ft from the end of the rod. Where is the center of oscillation? What is its period when the rod swings as a compound pendulum?

733. If the rod in Problem 732 starts from rest with its center of gravity directly above the pin and then swings freely in the vertical plane, determine the maximum values which α and ω will attain.

734. An irregular shaped steel bar is supported by a frictionless horizontal pin which passes through the bar at a point 3 ft from the center of gravity of the bar. The bar weighs 200 lb and oscillates 26 times per minute when set in motion. Determine the moment of inertia of the bar with respect to an axis through its center of gravity and parallel to the axis of the supporting pin.

735. A 1,000-lb flywheel turning 120 rpm is carried by a horizontal shaft. Bearing A is 18 in. from the wheel and bearing B is 30 in. from it. If the center of gravity of the flywheel is eccentric 0.2 in., determine the maximum values attained by the bearing reactions. *Ans.* $A = 676$ *lb;* $B = 406$ *lb.*

736. A weight of 100 lb is suspended from a weightless vertical coil spring whose scale is 50 lb per in. If the weight is pulled down 4 in. from its static position and released, determine: (*a*) the maximum velocity of the weight, (*b*) its maximum acceleration, and (*c*) its period of vibration.

737. A weight W is suspended from a weightless vertical coil spring. When the weight is set in motion, it attains a maximum velocity of 10 ft per sec and a frequency of 2 vibrations per sec. Determine (*a*) the amplitude, (*b*) the scale of the spring, and (*c*) the weight W.

738. A weight is suspended from the lower end of a vertical weightless coil spring. The weight is then pulled down 2 in. and released, causing it to vibrate 80 times per min. Determine (*a*) the displacement, (*b*) the velocity, and (*c*) the acceleration 2 sec after release.

739. Determine the amplitude and frequency of a simple harmonic motion, if the velocity is 8 ft per sec when the displacement from the center of the path is 6 in. and the velocity is 6 ft per sec for a displacement of 8 in.

740. A light stiff beam deflects 1 in. when a load of 2,000 lb is placed at the middle of the beam. What is the period of vibration when the beam is so loaded?

741. If the 2,000-lb load in Problem 740 is an electric motor whose rotor is slightly out of balance, at what speed would it be dangerous to operate the motor? Explain why.

742. A, B, and C are the centers of gravity of three weights revolving in the same plane about a point O. $OA = 18$ in., $OB = 25$ in., $OC = 15$ in. Angle AOB is 90°, and angle BOC is 120°. If the weight C is 50 lb, determine the amounts of A and B for complete running balance. *Ans.* $A = 36.1$ *lb;* $B = 15$ *lb.*

743. In Fig. 520, A and B are the centers of gravity of the halves of a generator rotor. The half A weighs 1,000 lb and is eccentric 0.03 in. Part B weighs 1,200 lb and is eccentric 0.06 in. Determine the weights which must be placed at C and D to completely balance the rotor.

744. If the rotor of Problem 743 is operated without balancing, determine the kinetic reactions at the bearings E and F for the position shown in Fig. 520. The rotor makes 1,800 rpm.

745. In Fig. 521 the shaft has a speed of 100 rpm. The rotating masses, which may be considered concentrated at the pin C, weigh 100 lb. These are

to be balanced by weights added to the flywheels. Determine the kinetic reactions at the bearings A and E without balancing weights. Determine the necessary balancing weights acting at a 12-in. radius in planes B and D.

746. In Fig. 522, EF is a wheel keyed to the vertical shaft G. The shaft and wheel are turning at 45 rpm. A 50-lb cylinder, with a pin connection at B, is carried by the wheel. Determine the tension in the cord CD. *Ans. 35.9 lb.*

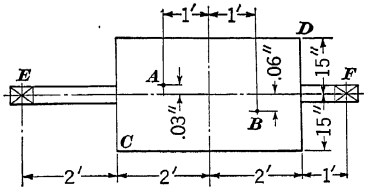

FIG. 520

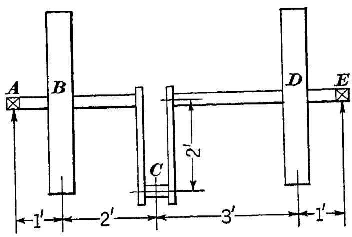

FIG. 521

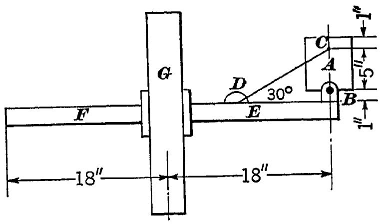

FIG. 522

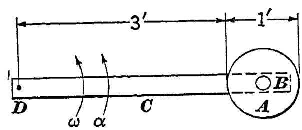

FIG. 523

747. A torsional pendulum has a moment of inertia of 0.6 ft-lb-sec^2 with respect to the axis of the supporting wire. It oscillates 100 times per min. When another body is placed on the pendulum so that its center of gravity coincides with the axis of the wire, the pendulum vibrates 65 times per min. What is the moment of inertia of the added object?

748. In Fig. 523, A is a 20-lb disk free to turn about the pin B; C is a weightless rod turning in the plane of the paper about the axis through D. If at a given instant the rod has an angular velocity of 30 rpm and an angular acceleration of π rad. per sec per sec, determine what will happen to the disk A.

749. When an automobile race driver goes around a curve, it is customary for him to increase his speed. Why is this done? What is apt to happen if the brakes are applied when entering a curve? Why?

CHAPTER 18

WORK, ENERGY, AND POWER

167. Work.—The popular conception of what constitutes work, and work as defined by the engineer and physicist, are not always the same. For example, a man standing perfectly still and supporting a 100-lb weight becomes fatigued. Therefore, according to the popular idea, he is doing work. According to the physicist and the engineer, however, no work has been done by the man.

Work, as defined by the engineer, is a mechanical process and therefore implies motion (a force moving through a distance). The man in the example above does no work so long as he does not move the 100-lb weight. If he lifts the weight, he does a relatively large amount of work against gravity; but, if he holds the weight in a given position and moves along a level floor, he does relatively little work on the weight. The work done then is only the small amount which is necessary to overcome the air resistance of the weight.

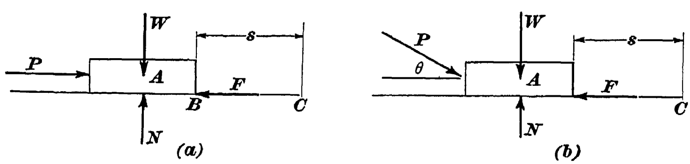

FIG. 524

When a constant force acts on a body and causes the body to move in such a manner that the displacement of the point of application of the force has a component in the direction of the force, mechanical work has been done. The measure of the amount of work done can best be illustrated by two simple examples.

In Fig. 524 (a) the block A is moved from B to C, a distance s along the plane. Neither of the forces W and N does work because its point of application has no displacement in the direction of its line of action. The constant force P does positive work equal to $P \times s$. Force F does negative work equal to $F \times s$.

In Fig. 524 (*b*) forces W and N again do no work because their points of application receive no displacements in the directions of those forces. Force F does negative work equal to $F \times s$, as in the previous example. The work of force P may be expressed in two ways: as $P \times s \cos \theta$, which is the product of the force and the component of the displacement in the direction of the force; or as $P \cos \theta \times s$, which is the product of the component of the force in the direction of the displacement and the displacement.

In the foregoing discussion the terms positive work and negative work have been used. In general, when the displacement is in the same direction as the active component of the force, the work done is positive. If the displacement and the active component of the force have opposite directions, the work done is negative. A force which tends to increase the velocity of a body does positive work on the body; a force which tends to decrease the velocity of a body does negative work on the body.

If a weight is falling, the work done by the force of gravity is positive; if the weight is being lifted, the work done against the force of gravity is negative. The work done by a frictional force is generally negative, but is positive in a few cases where the frictional force may be used to drive the mechanism.

Work can be defined mathematically by the equation

$$\text{Work} = U = F \times s$$

where F is a constant force and s is the distance which the point of application of the force F moves in the direction of the force F.

When the force F is a variable quantity,

$$\text{Work} = U = \int F\, ds$$

where F must be expressed in terms of the displacement s and ds is assumed to be sufficiently small for F to be considered constant over the distance ds.

The unit of work is a compound unit which is determined by the units used to express force and distance. In the United States force is generally expressed in pounds and distance is expressed in feet or inches. Therefore, the unit of work becomes the foot-pound or the inch-pound. Work is a scalar quantity and is independent of the time.

168. Work Done By a System of Forces.—When several forces act on a body to produce a displacement, the work done may be determined in the following manner.

(*a*) The resultant of the system of forces may be found. Then the product of the component of this resultant in the direction of the displacement and the displacement is the work done by the system.

(*b*) The work done by each of the several forces of the system may be determined. The algebraic sum of these works is then the resultant work of the system.

EXAMPLE

A body which weighs W lb is to be elevated a given distance against the force of gravity. Determine the work done.

If the body is divided into differential portions each of which weighs dW lb, each portion may be raised a different distance y from some horizontal datum plane. The work done on each portion will then be $dW\,y$, and the work required to lift the entire body can be determined from the following equation:

$$\text{Work} = \int y\,dW = W\,\bar{y}$$

In this equation $\bar{y}$ is the distance through which the center of gravity of the weight is raised above the datum plane.

PROBLEMS

750. What work is required to move a 100-lb weight at uniform speed up a 30° incline a distance of 50 ft, if $f = 0.2$? *Ans. 3,366 ft-lb.*

751. Assuming no losses, determine the amount of work required to fill a water tank 6 ft wide, 10 ft long, and 10 ft deep. The top of the tank is 75 ft above the surface of the stream from which the water is pumped. The pipe enters the tank at the bottom.

752. Solve Problem 751, if the tank is a hemispherical tank, the radius of which is 10 ft and the flat side of which is horizontal and on top.

753. A steel cable 150 ft long weighs 6 lb per ft and is supported by a pulley. If 100 ft of the cable hangs on one side of the pulley and 50 ft hangs on the other side, how much work will have to be supplied to exactly reverse the position of the cable?

754. A 3,000-lb automobile is driven up a 6% grade at a constant speed of 30 mi per hr. Frictional losses are 40 lb per ton. How many foot-pounds of work are done per min? What is the thrust of the tires parallel to the road?

755. A weight Q is moved a distance b along a horizontal plane by a weight P attached to the end of a cord passing over the pulley, Fig. 525. Determine the work done by the weight P in terms of b and the angles θ and ϕ.

756. Compute the least work required to move a 1,000-lb weight 20 ft up a 15° incline if the force is applied as in Fig. 526 and $f=0.3$. *Ans. 15,653 ft-lb.*

757. How much work must be done to build a brick chimney 200 ft high? The external diameter is 20 ft at the top and 30 ft at the base, and the internal diameter is 14 ft. Brick masonry weighs 100 lb per cu ft.

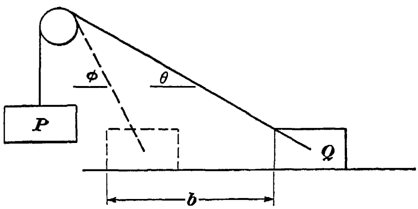

FIG. 525

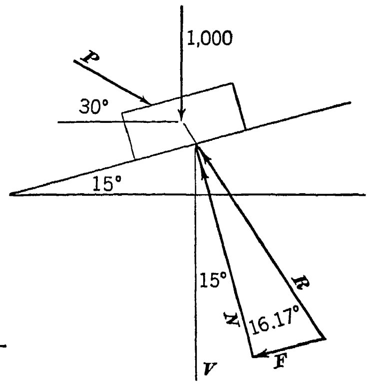

FIG. 526

169. **Energy.**—The energy possessed by a body is its capacity to do work. The body may possess this capacity to do work because of

(*a*) its position or form, or potential energy;

(*b*) its state of motion, or kinetic energy;

(*c*) its composition, thermal energy, chemical energy, electrical energy, etc.

A body possesses potential energy because of the position of the body relative to some other body. Water behind a dam or in an elevated tank has potential energy which is available for operating a wheel located at a lower level. The air in the tires of an automobile or any other reservoir possesses potential energy because of the manner in which it is confined. A compressed or elongated coil spring possesses potential energy because of the stresses set up in the material of the spring by the forced change in shape.

Potential energy of position is expressed by the following equation:

$$\text{P.E.} = \text{Available Work} = W\,h$$

where W is the weight of the body and h is its height above some given datum.

Kinetic energy is the energy of motion. It is the energy which enables a body in motion to do work against a resisting force F while its velocity is being reduced.

$$\text{K.E.} = \text{Work} = \int -F\, ds$$

Since $F = \frac{W}{g} a$,

$$\text{K.E.} = -\frac{W}{g} \int a\, ds$$

Also, $a\, ds = v\, dv$. Hence,

$$\text{K.E.} = -\frac{W}{g} \int_v^0 v\, dv = \frac{1}{2} \frac{W}{g} v^2$$

PROBLEMS

758. A 100-lb sand bag slides 40 ft down a smooth plane inclined at 30° with a smooth horizontal plane. The first bag strikes a 75-lb bag at rest on the horizontal plane. The two bags then move off at a speed of 20 ft per sec. Determine: (*a*) the loss in kinetic energy and (*b*) the loss in potential energy.

759. Water flows out of a horizontal pipe, whose cross-sectional area is $\frac{1}{2}$ sq ft, with a velocity of 10 ft per sec. The pipe is 10 ft above the discharge level of a water wheel. How much energy is theoretically available for operating the wheel?

170. Work and Kinetic Energy of Translation With Forces Constant.—In Fig. 527, W is any weight which is acted upon by a force P and a frictional resistance F. The weight starts from A with a velocity v_0 and attains a velocity v after having moved a distance s along the plane. The resultant force in the direction of motion is $(P - F)$. If this is used in the equation $\Sigma F = \frac{W a}{g}$,

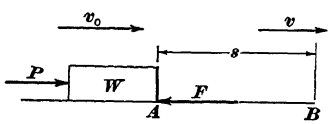

FIG. 527

$$P - F = \frac{W a}{g}$$

Since $v\, dv = a\, ds$,

$$P\, ds - F\, ds = \frac{W v\, dv}{g}$$

$$P \int_0^s ds - F \int_0^s ds = \frac{W}{g} \int_{v_0}^v v\, dv$$

$$P s - F s = \frac{1}{2} \frac{W v^2}{g} - \frac{1}{2} \frac{W v_0^2}{g}$$

On the left-hand side of this equation $P\,s$ is the work done in the direction of motion (positive work) by the force P and $F\,s$ is the resisting work (negative work) done by the frictional force F. The terms $\frac{1}{2}\frac{W\,v_0^2}{g}$ and $\frac{1}{2}\frac{W\,v^2}{g}$ are the kinetic energy of the body when it is at A and B.

If the preceding equation is transposed, it becomes a special form of the general energy equation, which is so often encountered in engineering calculations. Thus,

$$\frac{1}{2}\frac{W}{g}v_0^2+P\,s-F\,s=\frac{1}{2}\frac{W}{g}v^2$$

$$\text{I.K.E.}+\text{Pos. Wk.}-\text{Neg. Wk.}=\text{F.K.E.}$$

The fundamental principle of this equation is the conservation of energy. That is, for any given free body or force system, any body in motion possesses kinetic energy which is called initial kinetic energy. Any force which acts in a manner to increase this initial quantity of energy (increase the velocity) is doing positive work. A force which acts in such a way that it decreases this quantity of energy (decreases the velocity) is doing negative work. Any body in motion at the end of the cycle of operations possesses unused kinetic energy or final kinetic energy.

The general energy equation is not a vector equation; and, since it deals only with external forces, it can be applied to any system of connected bodies such as that in Fig. 530 or Fig. 531 as easily as to a single body such as that in Fig. 528. *The net work done on any system of connected bodies by external forces is equal to the change in kinetic energy of the system.*

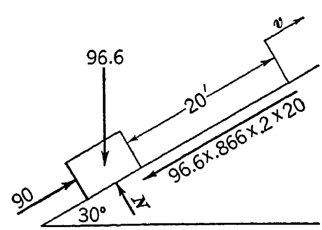

FIG. 528

EXAMPLE

As indicated in Fig. 528, a body weighing 96.6 lb is pushed up a 30° plane by a 90-lb force acting parallel to the plane. If the initial velocity of the body is 5 ft per sec and f for the plane is 0.2, what velocity will the body have after moving 20 ft?

$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

$$\frac{1}{2}\times\frac{96.6}{32.2}\times 25+90\times 20-96.6\times 20\times 0.5$$
$$-96.6\times 0.866\times 0.2\times 20=\frac{1}{2}\times\frac{96.6}{32.2}\,v^2$$

$$37.5+1{,}800-966-335=1.5\,v^2$$

$$v=18.9 \text{ ft per sec}$$

PROBLEMS

760. A 3,000-lb automobile has its speed increased from 10 mi per hr to 60 mi per hr in a distance of $\frac{1}{2}$ mile, while ascending a 2% grade. What constant thrust parallel to the surface of the road must the wheels exert? The total frictional resistance is 40 lb per ton. *Ans. 252.5 lb.*

761. A 50-lb weight slides 30 ft down a 30° slope; $f=0.3$ for the slope. If the point at which the body leaves the slope is 20 ft above the ground, with what velocity will the body hit the ground?

762. A 100,000-lb car is drawn up a 2% grade by a constant drawbar pull of 1,000 lb. If the car resistance is 8 lb per ton and the initial velocity is 30 ft per sec, how far will the car travel before its velocity is reduced to 10 ft per sec?

763. In the preceding Example, assume that the 90-lb force acts horizontally. Determine the velocity after the body moves 20 ft.

764. A 100-lb weight slides down a 30° plane for 50 ft and then up a 15° slope. If $f=0.3$ for both planes, how far up the 15° plane will the weight go? *Ans. 21.9 ft.*

765. If in Fig. 529 a weight starts down the incline at A with an initial velocity of 5 ft per sec and comes to rest on the level plane at B, determine the required distance d.

766. Determine the distance moved in 15 sec by the blocks in Fig. 530.

767. By the work and energy method, solve for the velocity of the 100-lb weight in Fig. 531, 10 sec after starting from rest. Determine the tension in the cord while the weights are in motion, assuming that $f=0.25$.

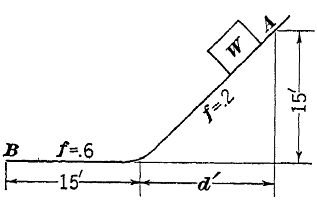

FIG. 529

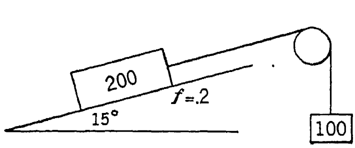

FIG. 530

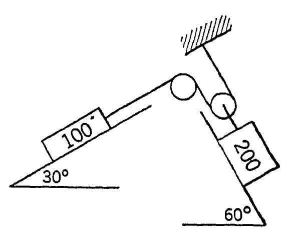

FIG. 531

171. Work and Kinetic Energy With Variable Forces.—In the equation given in Art. 170, or $P\int_0^s ds - F\int_0^s ds = \frac{W}{g}\int_{v_0}^{v} v\,dv$, the forces P and F may be constant or variable. If they are variable, each variable force must be expressed as a function of the distance s before the integration can be performed.

When a coil spring is compressed, the force required to deform the spring is directly proportional to the deformation. If C is the scale of the spring or the force required to deform the spring 1 in., then $C\,s$ is the force necessary to compress the spring s inches.

$$\int_0^s C\,s\,ds = \frac{C\,s^2}{2}\text{ in.-lb} = \frac{C\,s}{2}\times s = \text{Avg force}\times\text{dist.}$$

$$\int_0^s C\,s\,\frac{ds}{12} = \frac{C\,s^2}{24}\text{ ft-lb} = \frac{C\,s}{2}\times\frac{s}{12} = \text{Avg force}\times\text{dist.}$$

EXAMPLE

A 100-lb weight falls freely for 10 ft and then strikes a coil spring whose scale is 1,000 lb per in. How far will the coil spring be compressed?

$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

$$0 + 100\left(10 + \frac{s}{12}\right) - \frac{1{,}000}{2}\,s\,\frac{s}{12} = 0$$

$$1{,}000 + 8.33\,s - 41.66\,s^2 = 0$$

$$s^2 - 0.2\,s + 0.1^2 = 24 + 0.1^2$$

$$s - 0.1 = \pm 4.9$$

$$s = 5\text{ in.}$$

PROBLEMS

768. If the force required to stretch an elastic cord is directly proportional to the deformation and a force of 40 lb stretches the cord 5 in., how many foot-pounds of energy will be stored up in the cord when it is supporting a weight of 100 lb? *Ans. 52.1 ft-lb.*

769. If the cord in Problem 768 has 3 in. of slack when a 50-lb weight is released, how much will the cord be stretched when the weight is at its lowest position?

770. A 50-lb weight is projected upward with a velocity of 10 ft per sec. The weight falls back to a point 5 ft below its starting point where it strikes a coil spring. If the spring is compressed 9 in. by the falling weight, what is the scale of the spring?

771. A 10-lb weight is dropped from rest onto a coil spring whose scale is 100 lb per in. When the spring is compressed 6 in., the weight has a velocity

of 5 ft per sec. From what height above the uncompressed spring was the weight dropped?

772. A 200-lb weight starting from rest slides down a 30° plane for 10 ft where it strikes a spring. If the spring is compressed 15 in. by the weight and $f=0.2$, what is the scale of the spring?

773. A 100,000-lb gun has a recoil velocity of 10 ft per sec. If the recoil is resisted by a set of springs with a scale of 25,000 lb per in., how far will the gun recoil? *Ans. 12.2 in.*

172. Graphical Representation of Work.—In certain types of problems which involve work done by variable forces, it is difficult to express the force as a function of the distance; or, if it can be done, the expression is one which is very difficult to handle mathematically. Such cases can generally be simplified by the use of graphical methods.

Work is the product of the vector quantities force and displacement. It can therefore be represented by an area.

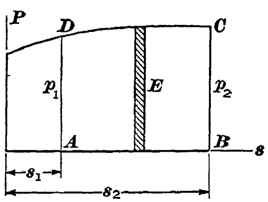

FIG. 532

In Fig. 532 displacements are measured along the s axis, while the corresponding ordinates represent the variable force which produces the displacement.

The work done during the displacement $(s_2 - s_1)$ is the area $ABCD$. This area can be obtained with a planimeter and converted into foot-pounds or inch-pounds of work, the units depending on the scale used in drawing the work diagram.

If a planimeter is not available, the area under the curve may be divided into narrow strips E, Fig. 532, of such small width that each may be taken as a rectangle or a trapezoid without introducing a large error. The area under the curve is then equal to the sum of the areas of the small strips.

Thus, if it is possible to obtain a sufficient number of values of a variable force, these values can then be plotted and a smooth curve can be drawn through the points. The work done by the variable force will then be represented by the area under the curve.

PROBLEM

774. The force acting on a given body varies according to the equation $F^2 = 4s$, where s is the displacement of the body in inches from a certain fixed point. If the initial value of the force is 10 lb and the body receives a displacement of 15 in., how much work has been done? Solve by calculus, and check by the narrow-strip method. *Ans. 170.6 in.-lb.*

173. Work Done in a Steam Cylinder.—Fig. 533 (*a*) is a diagrammatic sketch of a theoretical steam-engine cylinder. Fig. 533 (*b*) is the indicator diagram, which shows the variation in the steam pressure, in pounds per square inch, as the piston moves through a complete cycle.

From A to B steam is admitted to the cylinder at the full boiler pressure of p_1 psi; at B the cylinder port is closed and the steam in the cylinder expands until the point C is reached, when the exhaust port is opened. On the return stroke of the piston from D to E, the motion of the piston is opposed by a constant exhaust pressure of p_x psi.

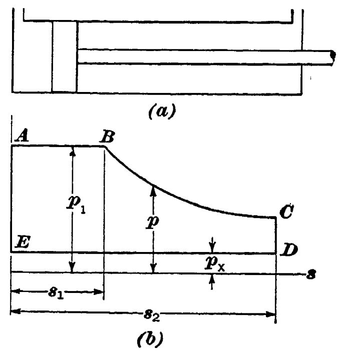

FIG. 533

The method of determining the work done during one complete cycle will be illustrated by the solution of the following example.

EXAMPLE

In Fig. 533 (*b*) assume that the boiler pressure is 100 psi, and cut-off is at $\frac{1}{4}$ stroke. The stroke is 24 in., and the back pressure is 20 psi. If the piston is 15 in. in diameter and the engine is single acting, how many foot-pounds of work are done in one complete cycle?

Area of piston $=176.7$ sq in.

Work from A to $B=100\times176.7\times0.5=8{,}835$ ft-lb

If it is assumed that from B to C the steam expands according to the law $P\ V=C$,

$$\text{Positive work from } B \text{ to } C=A\int_{s_1}^{s_2} p\,ds$$

Since $p_1\,v_1=p\,v$, $p=\frac{p_1\,v_1}{v}$; also, $\frac{v_1}{v}=\frac{s_1}{s}$ since the area of the piston is constant.

$$\text{Positive work from } B \text{ to } C=A\,p_1\,s_1\int_{s_1}^{s_2}\frac{ds}{s}=A\,p_1\,s_1\log_e\frac{s_2}{s_1}$$

$$=176.7\times100\times0.5\log_e 4=12{,}248 \text{ ft-lb}$$

$$\text{Negative work } D \text{ to } E=A\,p_x\,s_2=176.7\times20\times2=7{,}086 \text{ ft-lb}$$

$$\text{Net work}=8{,}835+12{,}248-7{,}068=14{,}015 \text{ ft-lb}$$

PROBLEM

775. If in the preceding Example the cylinder is placed in a vertical position and is made double acting, with a 2-in. diameter piston rod and cut-off at half stroke, how much work will be done during one revolution of the engine? The piston and rod weigh 300 lb. *Ans. 45,292 ft-lb.*

174. Work Done by a Force or Couple Applied to a Rotating Body.—In Fig. 534 the point of application A of the force F moves through a differential distance ds. The work done by the force F is the product of the component of F in the tangential direction, or $F\cos\phi$, and the distance ds.

$$\text{Work} = F\cos\phi\, ds = F\cos\phi\, r\, d\theta$$

But $F\cos\phi\, r = T$. If the body turns through an angle θ,

$$\text{Work} = T\int_0^\theta d\theta = T\,\theta$$

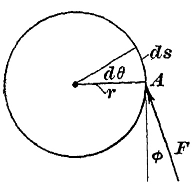

Fig. 534

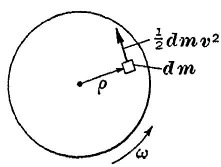

Fig. 535

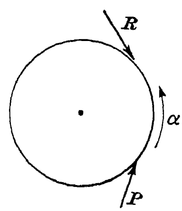

Fig. 536

175. Kinetic Energy of Rotation.—If the body in Fig. 535 is rotating with an angular velocity of ω rad. per sec, the kinetic energy of the particle $dm = \frac{1}{2}\, dm\, v^2$. The kinetic energy of the entire body will be

$$\text{K.E.} = \int \frac{1}{2}\, dm\, v^2$$

Since $v = \rho\,\omega$,

$$\text{K.E.} = \int \frac{1}{2}\, dm\, \rho^2\,\omega^2$$

But $\int \rho^2\, dm = I$, which is *the moment of inertia of the entire body about the axis of rotation.* Hence,

$$\text{K.E.} = \frac{1}{2}\, I\,\omega^2$$

176. **Work and Kinetic Energy of Rotation.**—In Fig. 536, P is a force producing a positive torque T_P in the direction of α; and R is a force producing a resisting or negative torque T_R. By Art. 152,

$$\text{Resultant Torque} = I\,\alpha$$
$$T_P - T_R = I\,\alpha$$
$$\int T_P\,d\theta - \int T_R\,d\theta = I\,\alpha \int d\theta$$
$$\omega\,d\omega = \alpha\,d\theta, \text{ or } \alpha = \frac{\omega\,d\omega}{d\theta}$$
$$\int T_P\,d\theta - \int T_R\,d\theta = I \int_{\omega_0}^{\omega} \omega\,d\omega$$

where ω is the final angular velocity and ω_0 is the initial angular velocity.

Since $\int T\,d\theta = \text{Work}$ (Art. 174),

$$\text{Positive Work} - \text{Negative Work} = \frac{1}{2}\,I\,\omega^2 - \frac{1}{2}\,I\,\omega_0^2$$

The right-hand side of this equation is the change in kinetic energy produced by the torques.

$$\frac{1}{2}\,I\,\omega_0^2 + \text{Pos. Wk.} - \text{Neg. Wk.} = \frac{1}{2}\,I\,\omega^2$$
$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

In this form the equation is similar to the energy equation for translation given in Art. 170.

EXAMPLE 1

A 1,000-lb cylinder, Fig. 537, is carried in bearings 6 in. in diameter. A rope wrapped around the cylinder has a 200-lb force at its free end. If $f = 0.1$ for the bearings, what is the speed of the cylinder, in rpm, after 20 ft of rope has been unwound?

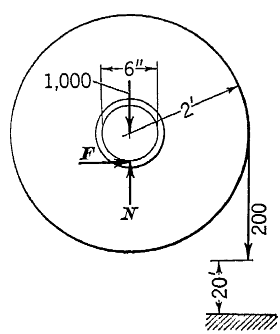

FIG. 537

$$\text{Positive work} = 200 \times 20 = 4{,}000 \text{ ft-lb}$$

$$\text{Negative work} = T\,\theta = \text{Torque} \times \text{angle}$$

Since $\theta = \frac{20}{2}$ rad.,

$$T\,\theta = (1{,}000 + 200)0.1 \times 0.25 \times \frac{20}{2} = 300 \text{ ft-lb}$$

Also,

$$\text{Negative work} = \text{Force} \times \text{distance}$$
$$= (1{,}000 + 200)0.1 \times 20 \times \frac{0.25}{2} = 300 \text{ ft-lb}$$

where the distance traveled by the tangential frictional force $= 20$ ft $\times$ ratio of the radii.

$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

$$0 + 4{,}000 - 300 = \frac{1}{2}\left(\frac{1}{2} \times \frac{1{,}000}{32.2} \times 2^2\right)\omega^2$$
$$3{,}700 = 31.1\,\omega^2$$
$$\omega = 10.9 \text{ rad. per sec}$$
$$\text{Speed} = \frac{10.9 \times 60}{2\pi} = 104.3 \text{ rpm}$$

EXAMPLE 2

The 250-lb weight in Fig. 538 (*a*) has a downward velocity of 30 ft per sec when the brake begins to act. If $f = 0.3$ for the brake, how far will the block travel before stopping? What is the tension in the rope while the brake is acting?

For the free body in Fig. 538 (*b*),

$$\Sigma M_O = 0$$
$$200 \times 4.75 - 0.75\,N - 0.3\,N \times 0.5 = 0$$
$$N = 1{,}055 \text{ lb}$$
$$F = 1{,}055 \times 0.3 = 317 \text{ lb}$$

For the system in Fig. 538 (*c*),

$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

$$\frac{1}{2} \times \frac{250}{32.2} \times 30^2 + \frac{1}{2} \times \frac{300}{32.2} \times 2.5^2 \times \left(\frac{30}{2}\right)^2 + 250\,d - 317\,d \times \frac{3}{2} = 0$$
$$d = 44.6 \text{ ft}$$

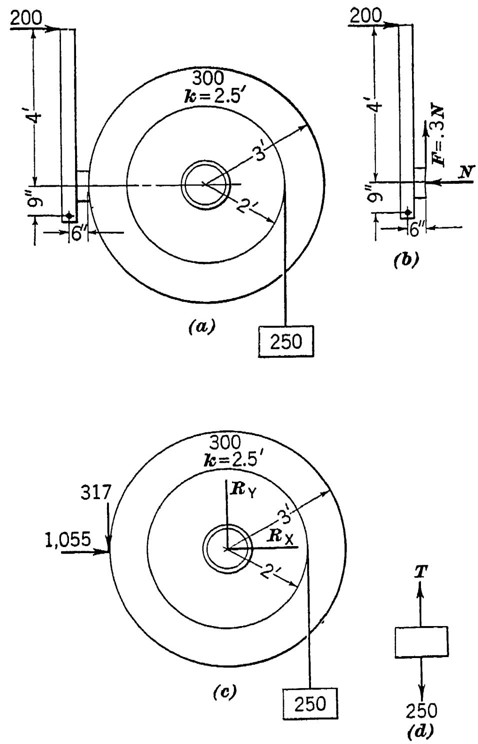

FIG. 538

For the free body in Fig. 538 (*d*),

$$\text{I.K.E.}+\text{Pos. Wk.}-\text{Neg. Wk.}=\text{F.K.E.}$$

$$\frac{1}{2}\times\frac{250}{32.2}\times 30^2+250\times 44.6-T\times 44.6=0$$

$$T=328 \text{ lb}$$

PROBLEMS

776. In the preceding Example 1, if the bearing friction is neglected and the 200-lb pull is changed to a 200-lb weight, what is the speed, in rpm, after the weight has moved 20 ft from rest? *Ans. 91.5 rpm.*

777. A 2,000-lb turbine rotor has a speed of 1,800 rpm when the steam is shut off. The bearings are 6 in. in diameter, with $f=0.012$. If windage resistance is neglected and $k=2.5$ ft, how long will the rotor continue to turn after the steam is turned off?

778. In Fig. 539, *A* is a 500-lb cylinder. Determine the velocity of the 100-lb weight after it has descended 30 ft from rest.

779. In Fig. 540, A is a mine-hoist drum. What normal pressure P must be applied to the brake B to bring the 500-lb weight to rest after it descends 50 ft? The drum is turning 80 rpm when the brake is applied.

780. Determine the angular velocity of the rod in Problem 680, Art. 155, by the work and energy method. *Ans. 4.71 rad. per sec.*

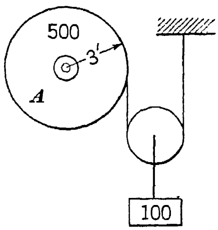

FIG. 539

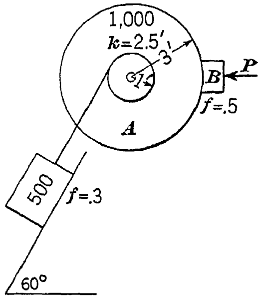

FIG. 540

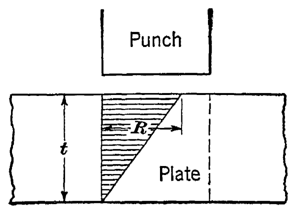

FIG. 541

781. Compute the angular velocity of the cylinder and rod in Problem 681, Art. 155.

782. The work diagram for a punch while punching steel plates is approximately represented by Fig. 541, where t is the thickness of the plate and R is the resistance offered to the passage of the punch through the plate. The punch has a solid disk flywheel 4 ft in diameter. If the speed of the flywheel is reduced from 120 rpm to 30 rpm while punching a 1-in. diameter hole in a plate $\frac{1}{2}$ in. thick, what is the weight of the flywheel? The ultimate shearing strength of the plate is 45,000 psi, and 10 per cent of the available energy is lost in the friction of the machine.

177. Power.—Power is the time rate of doing work or the measure of the work done in a given time. Mechanical work is generally expressed in foot-pounds. To measure power we must have a unit of power, or a rate of doing work which constitutes one power unit.

The generally accepted unit of power is the horsepower. A horsepower is 550 ft-lb of work per sec or 33,000 ft-lb per min. In terms of electrical units, 1 hp = 746 watts = 0.746 kw.

PROBLEMS

783. If a 7-ft diameter pulley has a belt with tensions of 1,000 lb and 250 lb on its tight and loose sides when the pulley has a speed of 225 rpm, what horsepower will the belt transmit to a generator? The generator efficiency is 85%. How many kilowatts will it supply?

784. A gravity hammer delivers 20 blows per min. The hammer weighs 500 lb and is lifted 4 ft for each blow. Determine the horsepower which must be supplied.

785. A 5,000-hp locomotive pulls a 1,200-ton train up a 2% grade. The frictional resistance of the train is 10 lb per ton. What is the speed of the train in mi per hr? How heavy a train could this locomotive pull at 30 mi per hr on a level track?

178. **Indicated Horsepower.**—The theoretical or indicated horsepower of an engine is obtained with the aid of an indicator diagram. The indicator diagram is a graphical picture of the pressure variation in a steam cylinder.

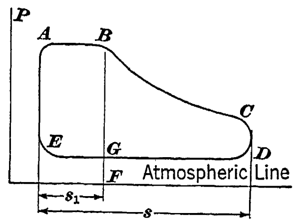

FIG. 542

The distance s in Fig. 542 represents the stroke of the piston to some scale, and any ordinate as BF is the unit pressure in pounds per square inch acting against the piston when the piston has moved a distance s_1 from the head-end of the cylinder. The ordinate GF is the back pressure acting against the piston when it reaches the same point on the return stroke.

If the area $ABCDE$ inside the curve, in square inches, is divided by the length, in inches, and the quotient is multiplied by the scale of the indicator spring, in pounds, the result is the average pressure or mean effective pressure, in pounds per square inch. If this mean effective pressure acted on the piston during the outward stroke only, it would produce the same effect as the positive pressure acting during the outward stroke and the back pressure acting during the back stroke.

If P is the mean effective pressure, in pounds per square inch, A is the area of the piston, in square inches, L is the length of the stroke, in feet, and N is the angular velocity, in rpm, then

$$\text{hp}=\frac{P\,L\,A\,N}{33{,}000}$$

This formula gives the horsepower developed by a single-acting engine, or an engine with steam on one side of the piston only. For a double-acting engine, the horsepower of each end must be computed to get the total horsepower of the engine.

PROBLEMS

786. The following data were taken from a 12″×24″ Corliss engine: Diameter of the piston rod, $2\frac{3}{16}$ in.; scale of the indicator spring, 80 lb; area of the head-end diagram, 2.04 sq in.; area of the crank-end diagram, 1.85

sq in.; and length of each diagram, 3.76 in. Determine the indicated horsepower of the engine if the angular velocity is 102.9 rpm. *Ans. 57.5 hp.*

787. An engine is 18 in.×24 in. and turns 150 rpm. Other data are: Diameter of piston rod, 3 in.; area of head-end card, 2.1 sq in.; area of crank-end card, 2.2 sq in.; length of each card, 3 in.; and scale of indicator spring, 100 lb. Compute the indicated horsepower.

179. Prony Brake.—A Prony brake is an apparatus used to determine the output or usable power developed by prime movers, such as steam engines, internal-combustion motors, electric motors, and water wheels.

The usable energy produced by the prime mover is transformed into frictional work at the surface of the brake wheel.

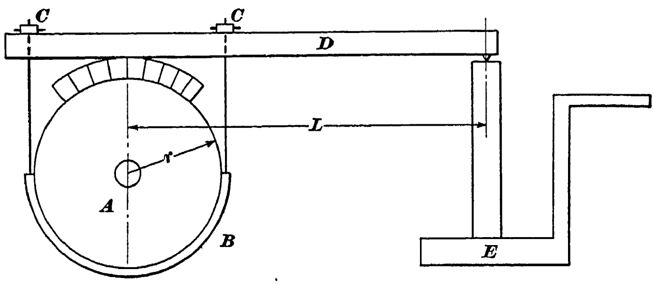

FIG. 543

A simple form of Prony brake is illustrated in Fig. 543. The pulley or brake drum A is keyed to the shaft of the prime mover. Around the drum is the adjustable brake band B. When the adjusting screw C is tightened, the friction between the brake band and the drum produces a torque which causes a pressure W to be exerted by the brake arm D on the scales E.

Taking moments with respect to the center of the brake drum gives the following relationship:

$$F \times r = W \times L \text{ or } F = \frac{W \times L}{r}$$

where F is the frictional force developed at the surface of the drum and W is the net reading of the scales after the static weight of the brake arm has been balanced.

If the brake drum is turning N revolutions per minute, the horsepower developed by the prime mover is given by the following equation, in which the length L must be expressed in feet:

$$\text{Brake hp} = \frac{2\,\pi\,r\,N\,F}{33{,}000} = \frac{2\,\pi\,r}{33{,}000} \times \frac{N\,W\,L}{r} = \frac{2\,\pi\,L\,N\,W}{33{,}000}$$

The efficiency of any prime mover is the ratio of the output of usable energy to the energy supplied to the machine. Efficiency is the energy conversion factor of the machine.

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{\text{bhp}}{\text{ihp}}$$

PROBLEMS

788. The engine in Problem 786, Art. 178, is fitted with a Prony brake for which L is 84 in., the tare reading of the scales is 35 lb, and the gross reading during the test is 390 lb. Determine the brake horsepower and mechanical efficiency of the engine. *Ans. 48.7 hp; 84.7%.*

789. The area of the head-end card for an 8″×12″ double-acting steam engine running at 227 rpm is 1.34 sq in.; the area of the crank-end card is 1.16 sq in.; the length of each card is 2.91 in.; and the scale of the spring is 60 lb. The piston rod is $1\frac{1}{2}$ in. in diameter. A Prony brake was attached to the engine. The tare for the brake was 40 lb and the gross reading of the brake scales was 110 lb. The length of the brake arm is 60 in. Determine the indicated horsepower, the brake horsepower, and the efficiency of the engine.

180. Water Power.—When water passes through a water wheel or turbine, the energy of the water is $\frac{1}{2}\,M\,v^2 = \frac{1}{2}\,\frac{W\,v^2}{g}$, where W is the weight of water flowing per second and v is the velocity of the water in feet per second.

If the hydraulic frictional losses of the pipe line are disregarded, the velocity of flow is given by the equation $v^2 = 2\,g\,h$, where h is the head or vertical fall in feet. Thus, if losses are disregarded, the horsepower output of the wheel is given by the following equation:

$$\text{hp} = \frac{1}{2}\,\frac{W\,v^2}{g \times 550} = \frac{1}{2}\,\frac{W \times 2\,g\,h}{g \times 550} = \frac{W\,h}{550}$$

This equation assumes that the water is at rest at the instant it begins its vertical fall and that it is possible to discharge the water from the wheel with zero velocity.

PROBLEMS

790. A hydraulic turbine is supplied with water under a head of 200 ft by a 4-ft diameter pipe. If the turbine has a mechanical efficiency of 90 per cent and absorbs 80 per cent of the available energy of the water, what horsepower is developed by the turbine? *Ans. 23,400 hp.*

791. Determine the horsepower required to cause a 6-in. diameter pipe to discharge 1,000 gal of water per min, if the discharge end of the pipe is 100 ft above the surface of the lake from which the water is pumped. A gallon of water weighs 8.33 lb.

792. A Pelton wheel has four $\frac{1}{2}$-in. diameter nozzles. Water is supplied under a head of 500 ft. If the wheel drives a generator, what is the output of the generator? The efficiency of the wheel is 85 per cent, and that of the generator is 90 per cent.

REVIEW PROBLEMS

793. An elevator weighing 1,000 lb is attached to the end of a 500-ft cable. If the cable weighs 2 lb per ft, how much work will be required to wind up the cable? What horsepower will be required if the cable is wound up in 45 sec? *Ans. 30.3 hp.*

794. A tank in the form of an inverted cone 10 ft in diameter and 16 ft high is filled with water by a pipe which enters the cone at the apex. The intake of the pump is 100 ft below the top of the tank. The pump efficiency is 85 per cent and the motor efficiency 90 per cent. If the tank is filled in 15 min, what horsepower is required?

795. Determine the least amount of work with which a 100-lb weight can be pushed up a 30° plane a distance of 50 ft. Assume that $f = 0.3$ for the plane.

796. Determine the work required in Problem 268 to lift the 10,000-lb weight 10 in.

797. A train of 50 cars, each weighing 150,000 lb, is being hauled up a $\frac{1}{2}$% grade 1,000 ft long, with a constant drawbar pull of 80,000 lb. If the initial speed is 6 mi per hr, what are the speed at the top of the grade and the maximum horsepower developed by the locomotive? The car resistance is 10 lb per ton. *Ans. 7.48 mi per hr; 1,594 hp.*

798. Determine the velocity of the 600-lb block in Fig. 544 after it has moved 20 ft from rest. What is the tension in the cord while the block is moving?

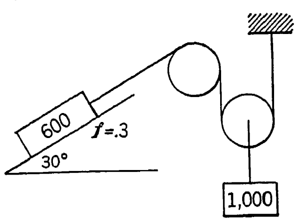

FIG. 544

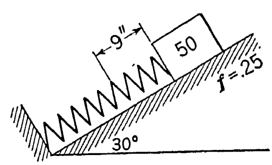

FIG. 545

799. Determine the velocity of the 1,000-lb weight of Problem 798 at an instant 10 sec after it starts from rest.

800. A freight train is going up a 1% grade at a speed of 15 mi per hr. If the rear car is uncoupled, with what speed will it reach a point 3,000 ft down the incline from the point of release? The frictional resistance of the car is 10 lb per ton.

801 A 150,000-lb freight car is going down a 2% grade at 20 mi per hr If the frictional resistance is 12 lb per ton and $f=0\ 3$ for the brake-shoes and wheels, what normal brake-shoe pressure is required at each of the eight wheels to stop the car in 2,000 ft?

802 A freight car weighing 150,000 lb has a velocity of 2 ft per sec when it strikes a bumping post Assuming that the drawbar spring takes all of the compression and the spring is compressed 3 in, determine the scale of the spring

803 A 100,000-lb freight car is switched up a 2% grade at 15 mi per hr Car resistance is 10 lb per ton The car is brought to rest at the top of the grade by striking a bumping post which has a 30,000-lb spring If the spring is compressed 8 in, how far up the grade did the car travel before striking the post?

804 The 50-lb weight, Fig 545, is pushed down the plane 9 in against a spring whose scale in 150 lb per in and is then released The spring acts on the weight only through the 9-in distance How far up the incline will the weight go?

805 If in Fig 545 the 50-lb weight slid down the 30° plane for 15 ft before striking the spring and then compressed the spring 6 in, what is the scale of the spring?

806 The 50-lb weight in Fig 545 is at rest on the 30° inclined plane against a spring, which has a scale of 40 lb per in It is pushed down the incline an additional 10 in and is then released Where will it be when its velocity is 5 ft per sec?

807 If in Fig 540 and Problem 779 the 500-lb weight is replaced by a 3,000-lb car which has a frictional resistance of 20 lb per ton, and the car is to be brought to rest by a normal pressure of 2,000 lb applied to the brake-shoe, how far from the top must the power be shut off? The car is ascending with a velocity of 20 ft per sec when the brake is applied? *Ans* 10 2 ft

808 What is the velocity of the 200-lb weight in Fig 546 after it has moved 10 ft from rest?

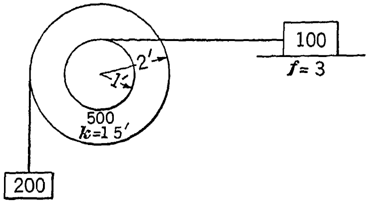

Fig 546

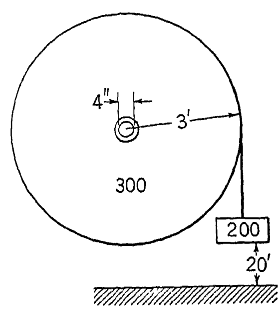

Fig 547

809 If in Fig 546 the 200-lb weight is supported by a 60° inclined plane for which $f=0\ 2$, what is the velocity of the 100-lb weight after it has been in motion for 10 sec?

810 How many turns will the 300-lb cylinder in Fig 547 make, if it starts from rest in the position shown and $f=0\ 15$ for the unlubricated 4-in diameter bearings?

811 Determine the velocity of the 161-lb weight in Fig 548 after it has moved 10 ft from rest

812 If in Fig 549 the weights are at rest in the position shown, how many rpm is the 128 8-lb cylinder making when the weights are at the same elevation?

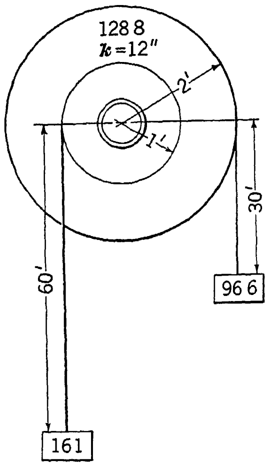

FIG 548

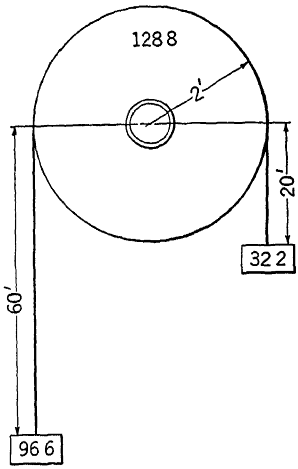

FIG 549

813 A slender symmetrical rod 4 ft long is mounted on horizontal frictionless bearings at a point 1 ft in from the end of the rod If the rod is released when in a horizontal position, what angular velocity will it have after it has moved 120° from the starting position?

814 In Fig 550, $f=0\ 5$ for the brake, and the 500-lb weight has a downward velocity of 10 ft per sec when the brake begins to act What is the velocity of the weight after moving 10 ft with the brake acting?

815 In Fig 551 the 800-lb weight has an upward velocity of 30 ft per sec when the brake begins to act How far will the weight move before coming to rest, if $f=0\ 3$ for the brake?

816 Determine the elongation of the spring in Fig 552 when the 200-lb weight is brought to rest after the system is released

817 A power shear is used to cut a sheet of steel $\frac{1}{2}$ in thick and 36 in wide The ultimate shearing strength of the plate is 50,000 psi Determine the weight which the flywheel must have if, while cutting the sheet, the speed of the flywheel is reduced from 120 rpm to 60 rpm The flywheel is a solid disk 5 ft in diameter, and 10 per cent of the available energy is lost in friction Assume the blade edge to be parallel to the surface of the sheet *Ans 3,610 lb*

818 What horsepower must a 3,000-lb automobile develop in order to go up a 10° slope at 30 mi per hr? The frictional resistance of the car is 40 lb per ton

819 A locomotive pulls a train of ten cars, weighing 100,000 lb each, up a 1% grade at 20 mi per hr The frictional resistance of the cars is 15 lb per ton What is the horsepower of the locomotive?

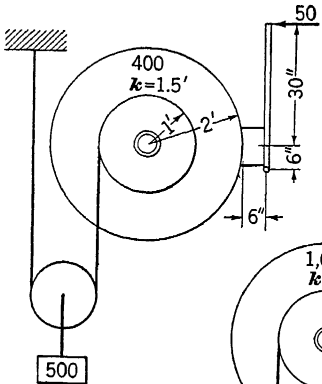

FIG. 550

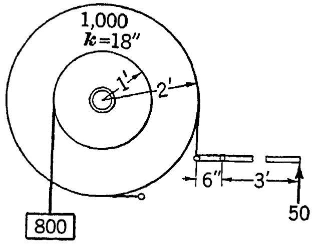

FIG. 551

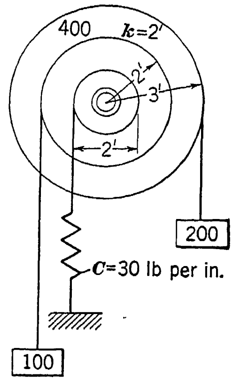

FIG. 552

820. A locomotive is developing 2,000 hp when traveling 30 mi per hr. What drawbar pull is it developing?

821. A 4,000-lb automobile increases its speed from 15 mi per hr to 45 mi per hr in 2 min 30 sec, while going up a 4% grade. If the car resistance is 125 lb and the acceleration is constant, what maximum horsepower is developed by the car?

822. A belt passes over a pulley 3 ft in diameter. The pulley is attached to a motor which is turning 300 rpm. If the tension on the tight side of the belt is 600 lb and that on the slack side is 200 lb, what horsepower is the belt transmitting? If the motor has an efficiency of 85 per cent, how many kilowatts are being furnished to the motor?

823. An electric mine hoist raises a load of 30,000 lb vertically. If the car attains a speed of 15 ft per sec in a distance of 20 ft, what maximum power, in kilowatts, is supplied to the motor? The over-all efficiency of the hoist is 70%, and the frictional resistance of the car is 10 lb per ton.

824. A fire engine takes water from a lake 12 ft below the engine and delivers it through a 2-in. diameter nozzle 20 ft above the engine with a velocity of 200 ft per sec. Determine the horsepower developed by the engine.

825. A bucket belt conveyor lifts 2 tons of ore per min to an elevation of 75 ft. The efficiency of the conveyor is 65% and that of the motor is 85%. How many kilowatts will be required?

826. A belt which passes over a pulley 4 ft in diameter is transmitting 100 hp when the speed of the pulley is 200 rpm. If the angle of contact of the belt with the pulley is 180° and $f=0.5$, what are the belt tensions when the belt is about to slip?

827. A 2-in. diameter nozzle discharges 20 cu ft per min at an elevation of 285 ft above the pump, which is placed 15 ft above the reservoir. How many kilowatts must be supplied to the pump motor if the efficiencies of the motor and pump are 90 and 85 per cent?

828. If 400 cu ft of water per sec passes through a turbine which has an efficiency of 85 per cent and the water is supplied under a head of 60 ft, what horsepower will the turbine develop? *Ans. 2,320 hp.*

829. The car in Fig. 553 is descending with a velocity of 30 mi per hr when the brake is applied. If $f=0.5$ for the brake, the car resistance is 20 lb per ton, and the car is to come to rest after traveling 50 ft, what force P must be applied to the brake lever?

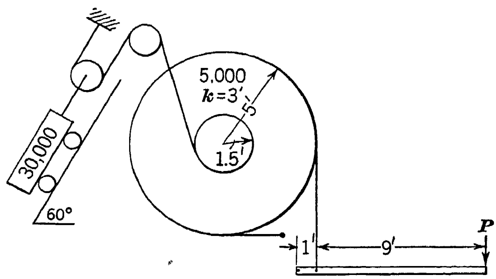

FIG. 553

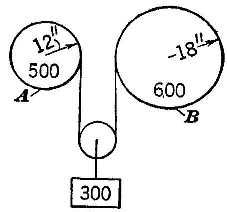

FIG. 554

830. In Fig. 554, A and B are solid cylinders which are free to turn about horizontal axes. Determine the velocity of the 300-lb weight after it has descended 20 ft from rest. Compute the tension in the cord while the weight is descending. How much rope will unwind from each of the cylinders?

CHAPTER 19

PLANE MOTION OF A RIGID BODY

181. Plane Motion of a Rigid Body.—When a rigid body executes plane motion, the center of gravity of the body moves in a given plane, the plane of motion. Each and every other particle of the body either moves in this given plane or in a plane which is parallel to the given plane.

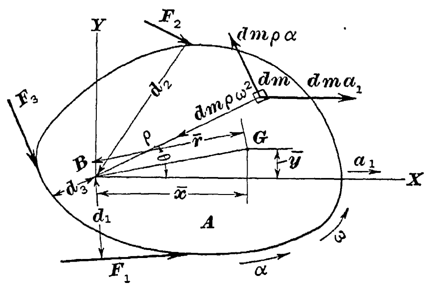

Fig. 555

When a rigid body, moving with plane motion, travels from one position to another, it does so by a simple translation or by a combination of translation and rotation (see Art. 134). If the motion is a simple translation, any line drawn on the body in the plane of motion will always be parallel to its original position. If the motion is a combination of translation and rotation, the succeeding positions of the line, as the body moves along its path, will not be parallel.

182. General Equations of Plane Motion.[1]—In Fig. 555, A is any rigid body which is moving with plane motion under the action of the unbalanced forces F_1, F_2, and F_3.

At any given instant, the forces F_1, F_2, and F_3 cause the entire body to move in such a manner that any given particle, such as B, has an absolute acceleration a_1 to the right along any line BX and every other particle, such as dm, rotates around B with an angular acceleration α and an angular velocity ω.

[1] In this discussion of plane motion the same conditions of symmetry are assumed as in Art. 151.

Since the particle of mass dm is rotating around B with an angular acceleration, it will receive accelerations in three directions (see Art. 137). Therefore, there are three effective forces acting on the particle. The forces $dm\ a_1$, $dm\ \rho\ \omega^2$, and $dm\ \rho\ \alpha$ are shown in Fig. 555.

If ΣF is the resultant of all such external forces as F_1, F_2, and F_3, then ΣF_x and ΣF_y will represent the X and Y components of ΣF.

According to the D'Alembert Principle, the resultant of all the effective forces acting on all the particles of the body is equal to the resultant of all the external forces such as F_1, F_2, and F_3. Therefore, the summation of the components of the external forces along any line is equal to the summation of all the effective forces along the same line.

$$\Sigma F_x = \int dm\ a_1 - \int dm\ \rho\ \alpha \sin\theta - \int dm\ \rho\ \omega^2 \cos\theta$$

$$= a_1 \int dm - \alpha \int dm\ \rho \frac{y}{\rho} - \omega^2 \int dm\ \rho \frac{x}{\rho}$$

$$= M\ a_1 - M\ \bar{y}\ \alpha - M\ \bar{x}\ \omega^2 \tag{1}$$

$$\Sigma F_y = \int dm\ \rho\ \alpha \cos\theta - \int dm\ \rho\ \omega^2 \sin\theta$$

$$= \alpha \int dm\ \rho \frac{x}{\rho} - \omega^2 \int dm\ \rho \frac{y}{\rho}$$

$$= M\ \bar{x}\ \alpha - M\ \bar{y}\ \omega^2 \tag{2}$$

$$\Sigma M_B = \Sigma(F\ d) = \int -dm\ a_1\ y + \int dm\ \rho\ \alpha\ \rho$$

Since $\int \rho^2\ dm = I$,

$$\Sigma M_B = -M\ a_1\ \bar{y} + I_B\ \alpha \tag{3}$$

Equations (1), (2), and (3) are the general equations of plane motion.

The solutions of many problems can be simplified by shifting the axes of reference. The problem under consideration is one of these cases. If the center of gravity is taken as the point of reference instead of the point B, the quantities $\bar{x}$ and $\bar{y}$ become zero (Art. 85), and the equations reduce to

$$\Sigma F_x = M\ \bar{a}$$
$$\Sigma F_y = 0$$
$$\Sigma M_G = I_G\ \alpha$$

In these equations, $\bar{a}$ is the absolute acceleration of the center of gravity and I_G is the moment of inertia of the body with respect to an axis through the center of gravity and perpendicular to the plane of motion.

Thus, in plane motion the center of gravity of a body receives the same acceleration it would receive if the system of forces acting were producing rectilinear motion only; in addition, the body receives an angular acceleration equal to that which it would receive if the body were turning around a fixed axis through its center of gravity under the action of the given system of forces.

Stated in another manner, the linear acceleration of the center of gravity of the body is dependent only on the mass of the body and the resultant component of the external forces in the direction of motion. The angular acceleration is dependent only on the moment of inertia of the body with respect to an axis through the center of gravity normal to the plane of motion and the resultant torque of the external forces with respect to this same axis.

Since the equations $\Sigma F_x = M\,\bar{a}$ and $\Sigma M_G = I_G\,\alpha$ are similar to the equations $\Sigma F = \frac{W}{g}\,a$ and Resultant Torque $= I\,\alpha$ of Arts. 143 and 152, either the effective force method or the inertia method of solution may be employed for problems involving plane motion.

183. Kinetic Energy During Plane Motion.—In Art. 135 it was shown that plane motion, at any given instant, is a simple rotation of the body about an axis known as the instantaneous axis or center.

The kinetic energy of the body about this axis is then K.E. $= \frac{1}{2}\,I_I\,\omega^2$, where I_I is the moment of inertia of the body with respect to the instantaneous axis. The value of I_I may be expressed in terms of the more easily determined moment of inertia with respect to the axis through the center of gravity of the body.

$$I_I = I_G + M\,\bar{r}^2$$

where $\bar{r}$ is the distance between the center of gravity and the instantaneous center.

$$\text{K.E.} = \frac{1}{2}\,(I_G + M\,\bar{r}^2)\,\omega^2 = \frac{1}{2}\,I_G\,\omega^2 + \frac{1}{2}\,M\,(\bar{r}\,\omega)^2$$

The quantity $\bar{r}\,\omega=\bar{v}$, where $\bar{v}$ is the absolute velocity of the center of gravity.

$$\text{K.E.}=\frac{1}{2}\,I_G\,\omega^2+\frac{1}{2}\,M\,\bar{v}^2$$

The kinetic energy of a body moving with plane motion is simply *the sum of its kinetic energy of translation and its kinetic energy of rotation about the center of gravity.*

184. **Free Rolling.**—Probably the most common example of plane motion is the rolling of a wheel, cylinder, or sphere along a plane. If there is no slipping or sliding, the motion is called free rolling or pure rolling. The solution of this type of problem will be illustrated by examples.

EXAMPLE 1

A 100-lb cylinder is rolled along a horizontal plane by a 30-lb force acting as indicated in Fig. 556 (*a*). Determine the linear and angular velocities of the cylinder after it has moved 30 ft from rest.

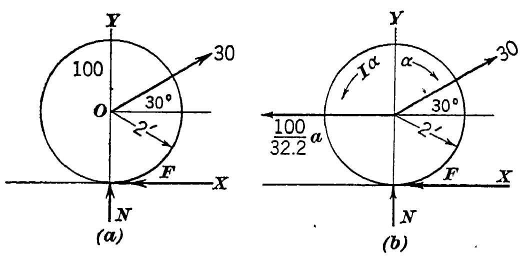

FIG. 556

Effective Force Solution.—The cylinder rolls because a frictional force F is induced at the surface of the plane.

$$\Sigma F_x=30\times 0.866-F=\frac{100}{32.2}\,a$$

$$25.98-F=3.105\,a \tag{1}$$

$$\Sigma F_y=N+15-100=0 \tag{2}$$

$$\Sigma M_O=2\,F=\frac{1}{2}\times\frac{100}{32.2}\times 4\,\alpha$$

For free rolling, $a=r\,\alpha$.

$$F=1.55\,a \tag{3}$$

Solving equations (1) and (3) gives $a=5.58$ ft per sec per sec.

$$F=8.65 \text{ lb}$$

$$v^2=v_0^2+2\,a\,s$$

$$v^2=2\times5.58\times30=334.8$$

$$v=18.3 \text{ ft per sec}$$

$$\omega=\frac{v}{r}=9.15 \text{ rad. per sec}$$

Inertia Solution.—For the conditions in Fig. 556 (*b*),

$$\Sigma F_x=0$$

$$30\times0.866-F-\frac{100}{32.2}\,a=0$$

$$\Sigma M_O=0$$

$$2F-\frac{1}{2}\times\frac{100}{32.2}\times4\,\alpha=0$$

The remainder of the solution is similar to that by the effective force method.

Work and Energy Solution.—In Fig. 556 (*a*),

$$\text{I.K.E.}+\text{Pos. Wk.}-\text{Neg. Wk.}=\text{F.K.E.}$$

$$30\times0.866\times30=\frac{1}{2}\times\frac{100}{32.2}\,v^2+\frac{1}{2}\left(\frac{1}{2}\times\frac{100}{32.2}\times4\right)\left(\frac{v}{2}\right)^2$$

$$v=18.3 \text{ ft per sec and } \omega=9.15 \text{ rad. per sec}$$

EXAMPLE 2

In Fig. 557 (*a*) a 500-lb hollow cylinder, 4 ft in outside diameter, has a rope wrapped around it; and the free end of the rope is

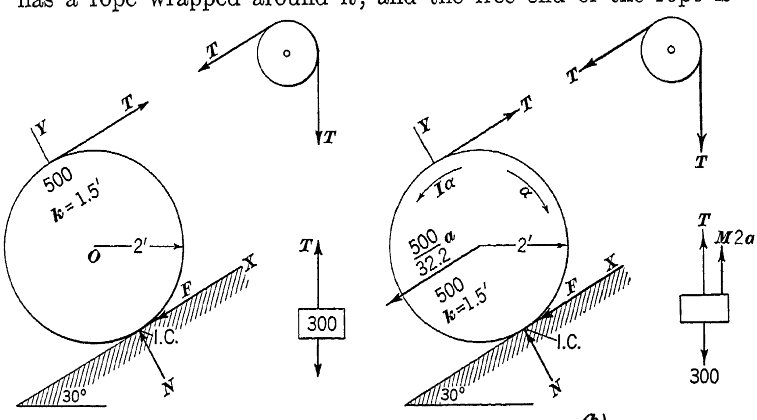

Fig. 557

attached to a 300-lb weight as indicated. If $k=1.5$ ft, determine the distance which the cylinder will roll in 10 sec from rest, the tension in the rope, and the amount and direction of the frictional force which acts on the cylinder.

Effective Force Solution.—Consider the cylinder in Fig. 557 (a) as a free body.

$$\Sigma F_x = T - F - 500\times 0.5 = \frac{500}{32.2}\,a$$

$$T - F - 250 = 15.5\,a \tag{1}$$

$$\Sigma M_O = 2\,T + 2\,F = \frac{500}{32.2}\times 1.5^2\,\alpha$$

Since $\alpha = \frac{a}{2}$ for rolling,

$$T + F = 8.74\,a \tag{2}$$

A vertical summation with the 300-lb weight as the free body gives:

$$300 - T = \frac{300}{32.2}\times 2\,a \tag{3}$$

Solving equations (1), (2), and (3) gives $a=5.68$ ft per sec per sec, $T=194$ lb, and $F=-144.2$ lb. The negative sign indicates that the force F in Fig. 557 (a) acts up the plane.

$$s = \frac{1}{2}\,a\,t^2 = \frac{1}{2}\times 5.68\times 100 = 284 \text{ ft}$$

Inertia Solution.—For the conditions in Fig. 557 (b),

$$\Sigma F_x = 0$$

$$T - F - 500\times 0.5 - \frac{500}{32.2}\,a = 0$$

$$\Sigma M_O = 0$$

$$2\,T + 2\,F - \frac{500}{32.2}\times 1.5^2\,\alpha = 0$$

$$\Sigma V = 0$$

$$T + \frac{300}{32.2}\times 2\,a - 300 = 0$$

Solve these equations as in the previous solution.

Work and Energy Solution.—In Fig. 557 (a), I. C. is the instantaneous center of the cylinder. Therefore, the distance

moved by the end of rope T (also the 300-lb weight) and its linear velocity and acceleration are twice those of the center of the cylinder. This is shown by the following equations, in which θ is the angular displacement of the cylinder and s is the tangential displacement.

$$s_T = 2\, r\, \theta \text{ and } s_0 = r\, \theta$$

The rolling cylinder has both kinetic energy of translation and rotation (Art. 183). The frictional force F does no work because it does not move.

$$\text{I.K.E.} + \text{Pos. Wk.} - \text{Neg. Wk.} = \text{F.K.E.}$$

$$0 + 300 \times 2\, d - 500 \times 0.5\, d = \frac{1}{2} \times \frac{300}{32.2} (2\, v)^2 +$$

$$\frac{1}{2} \times \frac{500}{32.2} v^2 + \frac{1}{2} \times \frac{500}{32.2} \times 1.5^2 \left(\frac{v}{2}\right)^2$$

$$600\, d - 250\, d = 18.65\, v^2 + 7.77\, v^2 + 4.38\, v^2$$

$$\text{But } d = \frac{v\, t}{2} \text{ or } v = \frac{2\, d}{10} = \frac{d}{5}$$

$$d = 284 \text{ ft}$$

Take the 300-lb weight as a free body.

$$300 \times 2 \times 284 - T \times 2 \times 284 = \frac{1}{2} \times \frac{300}{32.2} \left(\frac{2 \times 2 \times 284}{10}\right)^2$$

$$T = 194 \text{ lb}$$

PROBLEMS

831. Determine the time required for a 3,000-lb automobile to coast from rest down a 5% grade 1,000 ft long. The frictional resistance of the car is 100 lb. Each wheel is 28 in. in diameter and weighs 75 lb; and $k = 10$ in. for the wheels. *Ans. 62.5 sec.*

832. A 500-lb cylinder 1.5 ft in diameter rolls freely from rest down a 15° plane for 20 sec. How far will it roll? If the cylinder is just about to slip, what is the value of the coefficient of friction for the cylinder and the plane?

833. A sphere starts up a 30° incline with a linear velocity of 20 ft per sec. How far up the incline will it roll if there is no slipping?

834. A solid sphere rolls down a plane inclined at an angle θ with the horizontal. What is the minimum value of the coefficient of friction for free rolling.

835. A 100-lb cylinder is rolled along a horizontal plane by a 10-lb force applied at the end of the cord in Fig. 558. What are the linear acceleration of the cylinder and the value of the frictional force?

836. If $f=0.2$ for the plane and cylinder in Fig. 558, compute the maximum force which may be applied to the cord without causing slipping of the cylinder? What angular acceleration will the cylinder receive when the maximum force is acting?

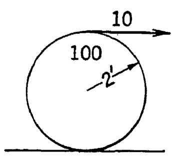

FIG. 558

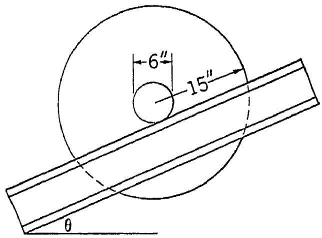

FIG. 559

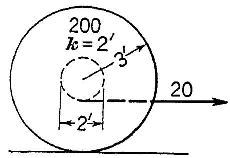

FIG. 560

837. In Fig. 559 a disk is mounted on a weightless shaft 6 in. in diameter. The shaft is supported by the two parallel rails. If $f=0.2$ for the shaft and rails and there is to be no slipping of the shaft on the rails, what maximum slope θ can the rails have?

838. Fig. 560 represents a 200-lb reel, such as is used to carry telephone cable. The outside diameter is 6 ft, the center portion on which the cable is wound is 2 ft in diameter, and $k=2$ ft. Determine the distance the reel will move in 10 sec. *Ans. 74.4 ft.*

839. Solve Problem 838 if the 20-lb force is directed upward at 60° with the horizontal and the diameter of the center portion of the reel is changed to 4 ft.

840. If a solid sphere and a cylinder having the same weight and diameter roll without slipping down a 30° plane, what is the minimum coefficient of friction for the plane? Which object will reach the end of the plane in the shorter time?

841. The reel of Problem 838 is placed on a 30° plane. The cable comes off the reel parallel to the plane at a point 180° from that shown in Fig. 560 and then passes over a pulley. A 100-lb weight is suspended from the free end of the cable. What is the velocity of the reel after it rolls 30 ft?

185. Plane Motion With Sliding.—When sliding occurs in plane motion, the relationship $a=r\,\alpha$ is not true. The angular acceleration of the body must then be determined from the equation

$$\text{Torque}=I\,\alpha$$

EXAMPLE

A 64.4-lb cylinder, 2 ft in diameter, moves from rest down a 30° plane, Fig. 561. If static $f=0.12$ and kinetic $f=0.10$ for the

plane, determine the linear and angular velocities of the cylinder after 10 sec.

Take the X axis parallel to the plane.

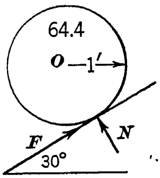

FIG. 561

$$\Sigma F_x = 64.4 \times 0.5 - F = \frac{64.4}{32.2} a$$

$$32.2 - F = 2\,a \qquad (1)$$

$$\Sigma F_y = N - 64.4 \times 0.866 = 0$$

$$N = 55.8 \text{ lb} \qquad (2)$$

If moments are taken with respect to an axis through O,

$$F \times 1 = \frac{1}{2} \times \frac{64.4}{32.2} \times 1^2\, \alpha$$

For free rolling, $\alpha = \frac{a}{1}$.

$$F = a \qquad (3)$$

By solving equations (1) and (3), it is found that $F = 10.73$ lb for pure rolling.

Available static force $= 55.8 \times 0.12 = 6.69$ lb, which is not equal to the amount required for pure rolling. The cylinder will slide and roll.

Available kinetic force $= 55.8 \times 0.10 = 5.58$ lb for sliding.

From equation (1),

$$32.2 - 5.58 = 2\,a$$

$$a = 13.31 \text{ ft per sec per sec}$$

For moments with respect to an axis through O,

$$5.58 \times 1 = \frac{1}{2} \times \frac{64.4}{32.2} \times 1^2\, \alpha$$

$$\alpha = 5.58 \text{ rad. per sec per sec}$$

$$v = v_0 + a\,t$$

$$v = 13.31 \times 10 = 133.1 \text{ ft per sec}$$

$$\omega = \omega + \alpha\, t$$

$$\omega = 5.58 \times 10 = 55.8 \text{ rad. per sec}$$

Many students will prefer the inertia method for solving this example and the following problems.

PROBLEMS

842. A 50-lb sphere 1 ft in diameter starts from rest and goes down a 30° plane 20 ft long. If static $f = 0.16$ and kinetic $f = 0.15$, what are the linear and angular velocities of the sphere at the bottom of the incline? *Ans. 21.85 ft per sec; 38.3 rad. per sec.*

843 A 400-lb cylinder 1 5 ft in diameter is placed on a 30° plane for which static $f=0\ 16$ and kinetic $f=0\ 15$ How far will it move in 10 sec? How many revolutions will it make?

844 Fig 562 shows a 600-lb cylinder 1 ft in diameter resting on two rails If a 100-lb force is applied to the end of the rope which is wrapped around the cylinder, determine the distance moved by the cylinder and the number of revolutions made by it in 10 sec Assume that static $f=0\ 15$ and kinetic $f=0\ 14$

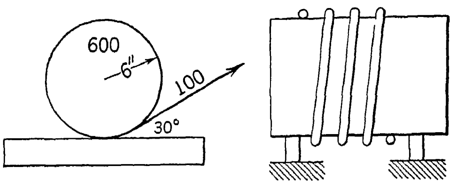

Fig 562

845 If the 20-lb force of Problem 838 is changed to 60 lb and static $f=0\ 10$ and kinetic $f=0\ 08$, what are the linear and angular velocities of the reel after 10 sec?

846 The reel of Problem 838 is placed on a 30° plane and the cable is led up the plane parallel to the surface and then over a pulley A 100-lb weight is suspended from the free end of the cable If static $f=0\ 17$ and kinetic $f=0\ 16$ for the plane, will the reel roll or slide? How far will it move in 10 sec? *Ans 161 5 ft*

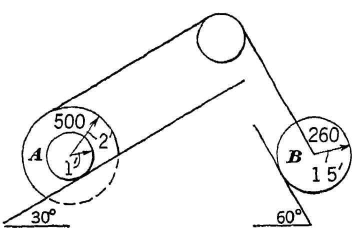

Fig 563

847 *A* and *B*, Fig 563, are solid cylinders Cylinder *A* has journals at each end which rest on supporting rails The mass of these journals is to be neglected Cylinder *A* rolls and cylinder *B* rolls and slides If kinetic $f=0\ 1$ for the 60° plane, determine the linear velocity of cylinder *A* after 10 sec

186 Reactions During Plane Motion—Many examples of plane motion occur in engineering where it is desirable to be able to solve for certain unknown forces or reactions, such as pin pressures on links, connecting-rods, eccentric rods, or side rods of locomotives

It has been shown in Art 134 that, when a body moves with plane motion, the motion at any given instant consists of a rotation, with or without an angular acceleration, about some axis which in turn is receiving a linear acceleration The motion is thus a superposition of a translation and a rotation

By the D'Alembert Principle, if the reversed resultant effective forces, or inertia forces, are added to the system of external forces acting on a body, equilibrium is established.

Because of the translation (Art. 143), there is a reversed resultant effective force, or inertia force, $M\,a$, acting through the center of gravity. Here, a is the linear acceleration of the axis about which the body is rotating. Therefore, a is also the linear acceleration of all parts of the body.

Because of the rotation (Arts. 153 and 154), there are two components of the reversed resultant effective force, or inertia force. One is $M\,\bar{r}\,\omega^2$, acting away from the axis of rotation along a line through the center of gravity; the other is $M\,\bar{r}\,\alpha$, acting perpendicular to $M\,\bar{r}\,\omega^2$ at a distance $\frac{k^2}{\bar{r}}$ from the axis, as demonstrated in Art. 154.

EXAMPLE 1

Assume that in Example 2, Art. 138, the linkage in Fig. 428 has the motion described because a 100-lb force acts on the sliding block along the line AC. Links AB and BC are slender rods, each weighing 30 lb. Determine the horizontal and vertical components of the pin reactions at B and C.

The solution of Example 2, Art. 138, gives the angular velocity of link BC as 4 rad. per sec, clockwise. The angular acceleration $\alpha=0$ and the linear acceleration of pin C (therefore, of all points on the link) is 48 ft per sec, horizontally to the left. The motion of the rod will then be taken as a translation plus a rotation at 4 rad. per sec about pin C.

Inertia Force Solution.—The link BC is shown as a free body in Fig 564.

$$M\,\bar{r}\,\omega^2=\frac{30}{32.2}\times1.5\times4^2=22.4$$

Since $\alpha=0$,

$$M\,\bar{r}\,\alpha=0$$

$$M\,a=\frac{30}{32.2}\times48=44.7$$

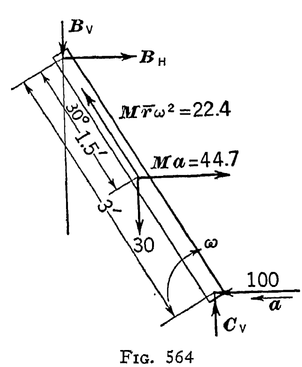

FIG. 564

$$\Sigma F_H=0$$

$$B_H+44.7-22.4\times0.5-100=0$$

$$B_H=66.5 \text{ lb} \longrightarrow$$

$$\Sigma M_B = 0$$

$$C_V \times 3 \times 0.5 - 30 \times 1.5 \times 0.5 + 44.7 \times 1.5 \times 0.866 - 100 \times 3 \times 0.866 = 0$$

$$C_V = 150 \text{ lb} \uparrow$$

$$\Sigma F_V = 0$$

$$-B_V - 30 + 22.4 \times 0.866 + 150 = 0$$

$$B_V = 139.4 \text{ lb} \downarrow$$

EXAMPLE 2

Determine the components A_N, B_N, and B_T of the pin reactions at A and B for the connecting-rod shown in Fig. 565. The crank OB is turning 120 rpm; the connecting-rod AB weighs 250 lb; $\bar{r} = 4$ ft; $I_A = 155.3$; $\frac{k^2}{\bar{r}} = 5$ ft; $l = 7$ ft; $\theta = 45°$; and $\phi = 7.25°$.

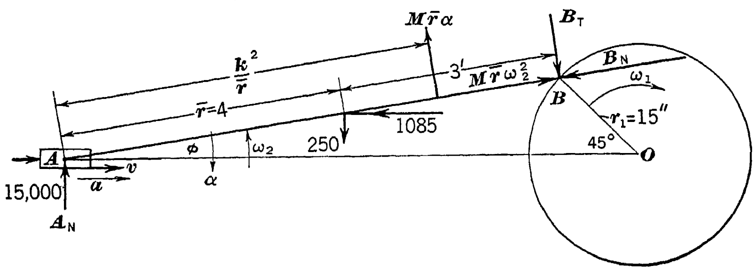

FIG. 565

$$\omega_1 = \frac{120 \times 2\pi}{60} = 12.56 \text{ rad. per sec}$$

$$v_B = r_1\, \omega_1 = 1.25 \times 12.56 = 15.7 \text{ ft per sec}$$

$$v_A = v_B \nrightarrow v_{\frac{A}{B}}$$

$$v_A = 15.7 \nrightarrow l\, \omega_2$$

This equation is solved graphically as in Fig. 566 (a) or by the sine law.

$$\frac{v_A}{\sin 52.25°} = \frac{l\, \omega_2}{\sin 45°} = \frac{15.7}{\sin 82.75°}$$

$$v_A = 12.5 \text{ ft per sec and } l\, \omega_2 = 11.2 \text{ ft per sec}$$

$$\omega_2 = 1.6 \text{ rad. per sec}$$

$$a_B = r_1\, \omega_1^2 = 1.25 \times 12.56^2 = 197.2 \text{ ft per sec per sec}$$

$$a_B = a_A \nrightarrow a_{\frac{B}{A}}$$

$$a_B = a_A \nrightarrow l\, \omega_2^2 \nrightarrow l\, \alpha$$

$$197.2 = a_A \nrightarrow 7 \times 1.6^2 \nrightarrow 7\, \alpha$$

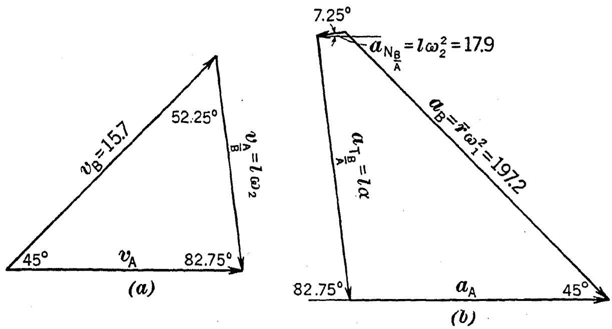

FIG. 566

This equation is solved graphically in Fig. 566 (*b*). It can be solved mathematically in the following manner:

By summing the vertical acceleration components,

$$197.2\times0.707=17.9\times0.126+7\,\alpha\times0.992$$
$$\alpha=19.74 \text{ rad. per sec}$$

By summing the horizontal acceleration components,

$$197.2\times0.707=-17.9\times0.992+138.2\times0.126+a_A$$
$$a_A=139.8 \text{ ft per sec per sec}$$

Since a_A is the linear acceleration of pin A, it is also the linear acceleration of every other point on rod AB.

$$M\,a=1{,}085$$
$$M\,\bar{r}\,\omega_2^2=79.5$$
$$M\,\bar{r}\,\alpha=612$$

In Fig. 565 the connecting-rod is shown with the three inertia forces added. This free body is in equilibrium.

$$\Sigma M_A=0$$
$$7\,B_T-612\times5+250\times4\times0.992-1{,}085\times4\times0.126=0$$
$$B_T=374 \text{ lb } \downarrow$$

$$\Sigma F_x=0$$
$$15{,}000-612\times0.126+374\times0.126-0.992B_N+79.5\times0.992-1{,}085=0$$
$$B_N=14{,}076 \text{ lb}\longleftarrow$$

$$\Sigma F_y=0$$
$$A_N-250+612\times0.992+79.5\times0.126-14{,}076\times0.126-374\times0.992=0$$
$$A_N=1{,}777 \text{ lb } \uparrow$$

PROBLEMS

848. Determine the horizontal and vertical pin reactions at pin A, Fig. 428 (a), for the conditions of Example 1, Art. 186. *Ans. 55.3 lb; 128.8 lb.*

849. A 100-lb slender uniform rod AB, Fig. 567, rests against frictionless surfaces at A and B. A force P causes the point A to move to the right at a constant speed of 10 ft per sec. Determine: (a) the angular velocity and acceleration of the rod and (b) the wall reactions at points A and B for the position shown.

850. The crank OB, Fig. 565, is turned clockwise 105°. Determine the components A_N, B_N, and B_T of the pin reactions at A and B if the 15,000-lb force is reduced to 5,000 lb.

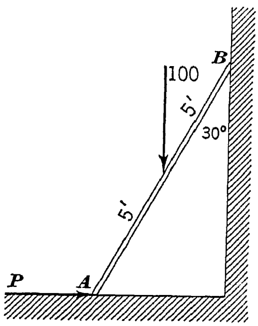

FIG. 567

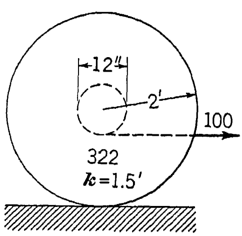

FIG. 568

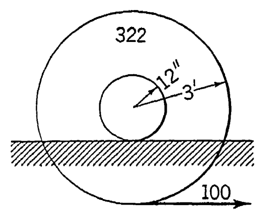

FIG. 569

REVIEW PROBLEMS

851. If a solid cylinder rolls freely down a plane which is inclined at an angle θ with the horizontal, what is the minimum value which the coefficient of friction f may have? *Ans.* $f = \frac{1}{3} \tan \theta$.

852. For the reel in Fig. 568 compute the minimum frictional force for rolling without slipping and the linear acceleration of the center of gravity.

853. Fig. 569 shows a cylinder with weightless hubs on each end. For rolling without slipping determine: (a) the direction of motion, (b) the required coefficient of friction, and (c) the rpm of the cylinder 10 sec after starting from rest.

854. Assume that a "yo-yo," such as children play with, weighs 4 oz. Let the yo-yo be considered to be a solid cylinder 4 in. in diameter, with the portion on which the string is wound 2 in. in diameter. If the free end of the string is held in the hand and the yo-yo is allowed to drop, how long will it take to fall 4 ft? What is its angular velocity, in rpm, after it has fallen 4 ft?

855. In Fig. 570 the 96.6-lb cylinder has the weightless inelastic cord wrapped around it. In the position shown, the cord has 5 ft of slack. What are the linear velocity and the angular velocity of the cylinder, in rpm, after it has fallen 15 ft from the position shown?

856. Determine the linear velocity and the rpm of the double pulley in Fig. 571 after it has fallen 10 ft from rest. *Ans. 11.34 ft per sec; 108.3 rpm.*

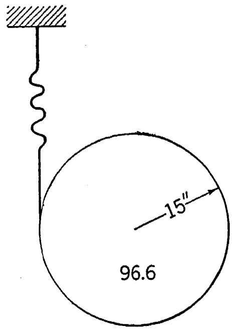

FIG. 570

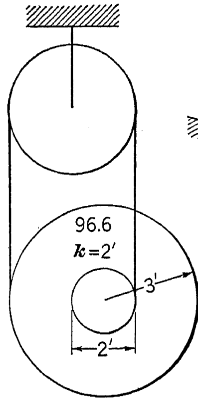

FIG. 571

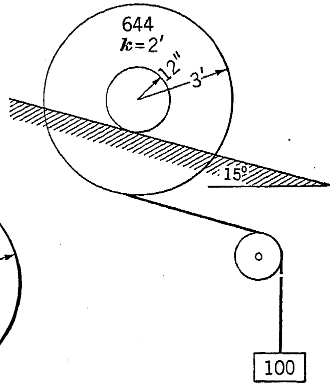

FIG. 572

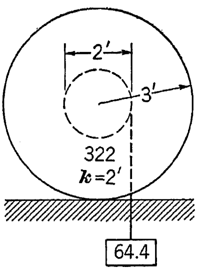

FIG. 573

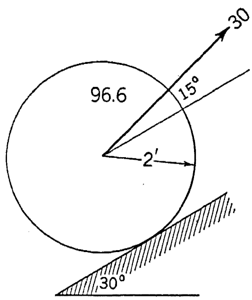

FIG. 574

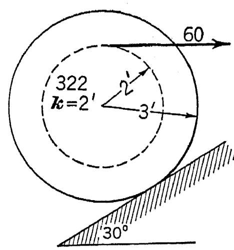

FIG. 575

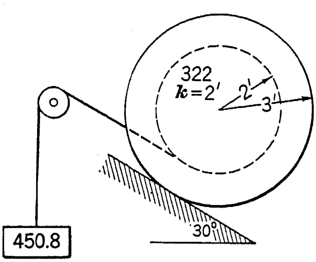

FIG. 576

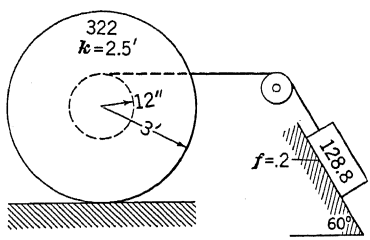

FIG. 577

857. Compute the velocity of the 100-lb weight in Fig. 572 after the drum has rolled 20 ft from rest without slipping.

858. Compute the angular velocity of the spool in Fig. 573 after it rolls 15 ft from rest without slipping. Determine also the magnitude and direction of the frictional force.

859. If the 96.6-lb cylinder in Fig. 574 starts from rest, how far will it move and what is the angular velocity after 10 sec? The coefficients of friction for the plane are static $f=0.12$ and kinetic $f=0.10$.

860. Compute the linear velocity of the spool in Fig. 575 at an instant 20 sec after it starts from rest, if static $f=0.18$ and kinetic $f=0.17$. *Ans. 113 ft per sec.*

861. Solve Example 2, Art. 184, if static $f=0.2$ and kinetic $f=0.15$ for the plane in Fig. 557 (*a*).

862. A solid sphere and cylinder, which have the same weight W and have the same diameter, are connected by a yoke in such a manner that both are free to roll. If they are placed on a plane inclined 15° with the horizontal, what is their velocity down the plane 10 sec after starting from rest? If the sphere is in front of the cylinder, what is the stress in the yoke? *Ans. 57.7 ft per sec; 0.0099 W.*

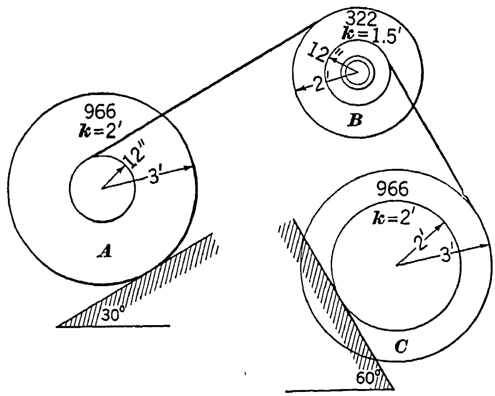

FIG. 578

863. Determine: (*a*) the time required for the spool in Fig. 576 to roll 15 ft from rest without slipping; (*b*) the final rpm.

864. The spool in Fig. 577 is to roll without slipping. Determine the minimum frictional force, if $f=0.2$ for both planes. Also compute the angular acceleration of the spool.

865. Determine the time required for the 128.8-lb weight in Fig. 577 to move 25 ft from rest.

866. Determine the linear velocity of reel C, Fig. 578, after reel A has moved 20 ft from rest. *Ans. 3.58 ft per sec.*

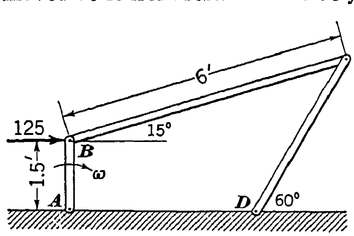

FIG. 579

867. At a given instant a four-link mechanism has the position shown in Fig. 579. Link AB has a constant angular velocity of 3 rad. per sec, clockwise. Link BC is a homogeneous slender rod which weighs 150 lb. Determine the horizontal and vertical components of the reactions at pins B and C if the horizontal reaction at pin B is 125 lb.

CHAPTER 20

IMPULSE AND MOMENTUM

187. **Definitions.**—The popular conception of an impulse is a large force acting for a short time, such as the blow of a hammer or the explosion of a charge of powder. According to Mechanics, linear impulse is simply the name which is given to the product of a force and the time during which the force acts. Impulse is thus just another measure of the effect of a force. Previously the effect of a force has been expressed in terms of the work done, which is the product of the force and the distance through which the force acts; or as the product of the mass acted upon and the acceleration produced by the force.

Since linear impulse is the product of a scalar quantity, time, and a vector quantity, force, linear impulse is a vector quantity and has the direction and position of the force. A constant force F acting during the time t produces a constant impulse, $F\,t$. If the force is variable, the impulse is given by $\int F\,dt$, where F must be expressed in terms of t.

The unit of linear impulse is the pound-second, which is the impulse produced by a force of 1 lb acting for 1 sec.

Linear momentum is measured in terms of the product of the mass of the body and its linear velocity. Since velocity is a vector quantity, linear momentum is represented by a vector which has the same direction and position as the velocity vector.

The unit of linear momentum is a mass of 32.2 lb moving with a velocity of 1 ft per sec.

$$\text{Unit of linear momentum} = \frac{W\,v}{g} = \frac{\text{lb}}{\frac{\text{ft}}{\text{sec}^2}}\,\frac{\text{ft}}{\text{sec}} = \text{lb} \times \text{sec}$$

188. **Relation of Linear Impulse to Linear Momentum.**—If a constant resultant force, or unbalanced force, F acts on a body with a mass M, then according to Newton's Second Law the body will receive a constant acceleration a.

$$F = M\,a$$

Since $a=\dfrac{dv}{dt}$,

$$F=M\,\frac{dv}{dt}$$

$$\int_0^t F\,dt=M\int_{v_0}^{v} dv$$

where v_0 is the velocity of the body when the force F starts to act and v is the velocity after the force has acted for t sec.

$$F\,t=M\,v-M\,v_0$$

If F is a variable force which can be expressed in terms of t, then $\int F\,dt$ can be integrated. If F varies in an unknown manner, it may be necessary to eliminate the quantity $\int F\,dt$ from two independent equations.

The relationship just derived may be stated as follows:

Resultant Linear Impulse = Change in Linear Momentum

The equation may also be transformed into an equation which is similar in form to the General Energy Equation of Art. 170.

Initial Linear Momentum + Positive Linear Impulse −
Negative Linear Impulse = Final Linear Momentum

When the equation is used in this form, any impulse which is in the direction of the initial momentum is a positive impulse and any impulse in the opposite direction is a negative impulse.

Since all terms in this equation are vector quantities, it is necessary that the vector relationship be maintained. This is in contrast to the situation in the General Energy Equation, where all terms are scalar quantities.

It will be observed that the acceleration does not appear in the impulse-momentum equations. They are therefore especially convenient for the solution of problems which do not require the determination of the acceleration or those in which the acceleration is a variable quantity.

EXAMPLE 1

A 100-lb weight starts down a 30° plane with an initial velocity of 10 ft per sec. What is the velocity of the weight after 10 sec, if $f=0.2$ for the plane?

Resultant Impulse Solution:

$$\Sigma F \text{ parallel to plane} = 100\times 0.5 - 100\times 0.866\times 0.2 = 32.68 \text{ lb}$$

$$\text{Resultant Impulse} = \text{Change in Momentum}$$

$$32.68\times 10 = \frac{100}{32.2}(v-0)$$

$$v = 105.3 \text{ ft per sec}$$

Since the initial velocity is 10 ft per sec, the final velocity is

$$10+105.2 = 115.2 \text{ ft per sec}$$

Solution by General Equation;

$$\text{I.L.M.}+\text{P.L.I.}-\text{N.L.I.}=\text{F.L.M.}$$

$$\frac{100}{32.2}\times 10+100\times 0.5\times 10-100\times 0.866\times 0.2\times 10=\frac{100}{32.2}v$$

$$v = 115.2 \text{ ft per sec}$$

EXAMPLE 2

A 200-lb weight is sliding to the right with a velocity of 40 ft per sec on a smooth horizontal plane, Fig. 580, when a 100-lb force directed to the left and 30° above the plane begins to act on the body. If the force acts for 20 sec, what is the velocity of the body?

$$\text{I.L.M.}+\text{P.L.I.}-\text{N.L.I.}=\text{F.L.M.}$$

$$\frac{200}{32.2}\times 40+0-100\times 0.866\times 20=\frac{200}{32.2}v$$

$$v = -238.5 \text{ ft per sec}$$

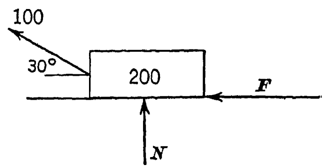

FIG. 580

The negative sign indicates that the body is moving to the left after the 20 sec.

EXAMPLE 3

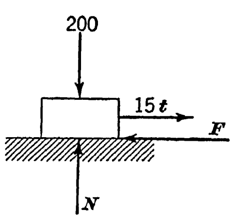

FIG. 581

A 200-lb block, Fig. 581, rests on a horizontal plane for which static $f=0.3$ and kinetic $f=0.25$. If a variable horizontal force $P=15\ t$ acts on the block for 15 sec, what velocity does the block attain?

The limiting static frictional force is

$$200\times 0.3 = 60 \text{ lb}$$

The time t for which the applied force must act before the block moves is found from the relation

$$60=15t$$

Hence,
$$t=4 \text{ sec}$$

$$\text{I.L.M.}+\text{P.L.I.}-\text{N.L.I.}=\text{F.L.M.}$$

$$0+\int_4^{15} 15\,t\,dt-200\times 0.25\times 11=\frac{200}{32.2}\,v$$

$$v=163.8 \text{ ft per sec}$$

PROBLEMS

868. A 100-lb body starts up a 30° plane ($f=0.2$) with a velocity of 8 ft per sec. How long must a 100-lb horizontal force act on the body to increase its velocity to 20 ft per sec? How far will the body travel while the force is acting? *Ans. 4.02 sec; 56.28 ft.*

869. If in Problem 868 the 100-lb force is changed to a force $F=80+20t$ acting parallel to and up the plane, what time will be required to attain the velocity of 20 ft per sec?

870. In Example 2, if the plane has a coefficient of friction $f=0.2$, what is the velocity of the block after 20 sec?

871. A 100,000-lb car A, coupled to a 60,000-lb car B, starts from rest on a 5% grade. The rolling resistance of each car is 10 lb per ton. The rear car A has its brakes set slightly so that they develop an additional resistance of 20 lb per ton. How long will it take the cars to attain a speed of 30 mi per hr? What is the pull on the coupler? How far will the cars travel before they attain this speed?

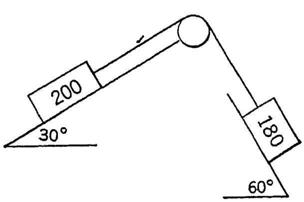

FIG. 582

872. If, in Fig. 582, $f=0.2$ for both planes, determine the time required for the weights to attain a velocity of 20 ft per sec. What is the tension in the cord? *Ans. 71.9 sec; 136.3 lb.*

873. If in Problem 868 the 100-lb force is changed to a force $F=80+20t$ acting parallel to and up the plane, when will the block attain a velocity of 20 ft per sec?

874. A 100-lb block has a velocity of 20 ft per sec to the right along a horizontal plane at the instant at which a horizontal force $P=6t+6$ directed to the left begins to act. If $f=0.3$ for the plane, what is the velocity of the block 4 and 10 sec after the variable force begins to act?

875. A 100-lb block rests on a horizontal plane for which $f=0.1$. If a variable horizontal force $P=3t-t^2$ is applied to the block, what is the velocity of the block after 10 sec? *Ans. 41.6 ft per sec.*

876. A machine gun fires 300 bullets per min. If each bullet weighs 1 oz and leaves the gun with a velocity of 2,000 ft per sec, what is the average pressure of the gun against its support?

189. Conservation of Linear Momentum.—From the equation $\int F\,dt = M\,v - M\,v_0$, Art. 188, it is evident that, if the resultant of all the external forces which act on any given body has no component along any given line, then the momentum of the body along that line will remain constant.

If two bodies with masses M_1 and M_2 collide, the first will exert a pressure on the second, and the second will exert an equal and opposite pressure on the first. Since these two equal and opposite forces act for the same length of time, their impulses must be equal and opposite. The net result is a zero impulse. Therefore, if there are no external forces acting on the two masses during the collison, there can be no change in linear momentum. From this discussion is developed the principle of conservation of linear momentum, which is stated as follows:

For any common or mutual action between two bodies, the total linear momentum before the action is equal to the total linear momentum after the action, when no external forces are acting. Stated in the form of an equation, this principle gives

$$M_1\,v_1 + M_2\,v_2 = M_1\,v_1' + M_2\,v_2'$$

where v_1 and v_2 are the velocities before the action; and v_1' and v_2' are the velocities after the action. It is generally convenient to give the velocity v_1 the positive sign. Velocities in the same direction as v_1 have the plus sign; velocities in the opposite direction have the negative sign.

Since work is done in deforming the bodies during the action, there must be a loss in kinetic energy. This is indicated by the increase in temperature of the bodies.

EXAMPLE

A 50,000-lb car traveling 6 mi per hr is shunted onto a side track where it meets a 100,000-lb car which is traveling 1 mi per hr in the opposite direction. When the cars meet, they are coupled together. What is the speed of the cars after they meet? Determine the loss of kinetic energy.

$$\frac{50{,}000}{32.2}\times 8.8 - \frac{100{,}000}{32.2}\times 1.46 = \frac{150{,}000}{32.2}\,v$$

$$v = 1.96 \text{ ft per sec}$$

The kinetic energies are:

$$\frac{1}{2}\times\frac{50,000}{32.2}\times 8.8^2 \quad = 60,120 \text{ ft-lb}$$

$$\frac{1}{2}\times\frac{100,000}{32.2}\times 1.46^2 = \quad 3,310$$

$$\overline{63,430} \text{ ft-lb}$$

$$\frac{1}{2}\times\frac{150,000}{32.2}\times 1.96^2 = 8,945 \text{ ft-lb}$$

Loss of kinetic energy $= 63,430 - 8,945 = 54,485$ ft-lb

PROBLEMS

877. A 10-lb shell is given a muzzle velocity of 1,400 ft per sec as it leaves a 6,000-lb gun. The gun recoils 6 in. against a coil spring. Determine the scale of the spring. *Ans. 337 lb per in.*

878. A 1-lb projectile which has a velocity of 2,000 ft per sec is fired into a 200-lb sand bag at rest on a plane inclined at 15° with the horizontal. If $f = 0.1$ for the plane and the direction of the bullet is parallel to the plane, how far up the plane will the bag move?

879. The 16-lb sledge in Fig. 583 strikes the 20-lb wood block which is at rest on the coil spring. The striking velocity of the sledge is 20 ft per sec and the block and sledge are assumed to remain in contact after striking. If the blow causes the spring to be compressed 2 in., what is the scale of the spring?

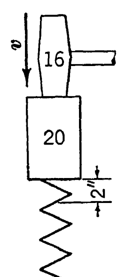

FIG. 583

880. A 180-lb man running with a horizontal velocity of 25 ft per sec jumps off a dock into a 300-lb boat which is moving toward him with a velocity of 10 ft per sec. What is the horizontal velocity of the boat after the man lands in it?

881. A 1-lb projectile is shot into a 100-lb sand bag. The bag is hanging from a rope 5 ft long. How high will the bag swing if the velocity of the projectile is 1,000 ft per sec?

882. A 60,000-lb car meets a 40,000-lb car which is moving in the opposite direction with a speed of 2 ft per sec. The two cars are coupled and then move with a speed of 8 ft per sec in the direction in which the first car was moving. What was the speed of the first car when it met the second?

190. Impact.—If the centers of gravity of two bodies, before collision, move along the same straight line and if the bodies are of such shape that the mutual pressures which they exert on each other during the collision act along the line connecting the centers of gravity, then the action is called direct central impact.

Observation tells us that, if two inelastic bodies meet in direct central impact, they will remain in contact after impact and will move off with a common velocity. If the bodies are elastic or

partially elastic, they will separate and each will have a different velocity after impact.

The equation developed in Art. 189 will not determine the velocities of two elastic or partially elastic bodies after impact because such bodies move off after impact with velocities which are dependent on the masses of the bodies and their elasticities. Newton was the first to observe that the relative velocity of two elastic or partially elastic bodies after impact may be determined by multiplying their relative velocity before impact by a factor which depends on the material of the bodies.

If the velocities before impact are v_1 and v_2 and the direction of v_1 is taken as the positive direction, the relative velocity before impact is $v_1 - v_2$. After impact, if the velocities are v_1' and v_2', the relative velocity is $v_2' - v_1'$.

For perfectly elastic bodies,

$$v_1 - v_2 = v_2' - v_1'$$

For partially elastic bodies,

$$(v_1 - v_2)\, e = v_2' - v_1'$$

where the quantity e, known as the *coefficient of restitution*, is an experimentally determined factor or ratio which depends on the material of the bodies. For perfectly elastic bodies, $e=1$; for partially elastic bodies, e is less than 1 and more than zero. The following are values of e for some of the more often used materials. For glass, $e=0.95$; for ivory, $e=0.89$; for steel, $e=0.55$; for cast iron, $e=0.50$; for lead, $e=0.15$. Other values may be found in the engineering handbooks.

The use of the preceding equation in connection with $M_1 v_1 + M_2 v_2 = M_1 v_1' + M_2 v_2'$ will now be illustrated.

The direction of v_1 is usually taken as the positive direction. Velocities in the opposite direction are then negative.

EXAMPLE

A 5-lb ball moving with a velocity of 10 ft per sec strikes a 10-lb ball moving in the opposite direction with a velocity of 1 ft per sec. If $e=0.6$, what is the velocity of each ball after impact?

$$M_1 v_1 + M_2 v_2 = M_1 v_1' + M_2 v_2'$$

$$\frac{5}{32.2} \times 10 + \frac{10}{32.2}(-1) = \frac{5}{32.2} v_1' + \frac{10}{32.2} v_2'$$

$$40=5\,v_1'+10\,v_2' \tag{1}$$

$$(v_1-v_2)\,e=v_2'-v_1'$$

$$[10-(-1)]\,0.6=v_2'-v_1'$$

$$6.6=v_2'-v_1' \tag{2}$$

Solving equations (1) and (2) gives

$$v_1'=-1.74 \text{ ft per sec and } v_2'=4.86 \text{ ft per sec}$$

PROBLEMS

883. A 10-lb ball moving with a velocity of 20 ft per sec strikes a 4-lb ball at rest. If $e=0.5$, what are the velocities of the two balls after impact? *Ans. 11.45 ft per sec; 21.45 ft per sec.*

884. A ball falls 20 ft and rebounds 8 ft from a hard floor. Determine the value of the coefficient of restitution.

885. An 8-lb steel hammer strikes a 5-lb steel ball at rest. If the velocities after impact are 7 and 25 ft per sec, respectively, and they are in the same direction, what was the striking velocity of the hammer and what is the value of e?

886. How high will the ball of Problem 884 rise on the third rebound?

887. A 40-lb ball moving with a velocity of 4 ft per sec strikes a 10-lb ball moving in the opposite direction with a velocity of 50 ft per sec. If $e=0.6$, what are the velocities after impact? *Ans. −13.28 ft per sec; 19.12 ft per sec.*

888. An 80-lb ball, moving with a velocity of 5 ft per sec to the right, strikes a 100-lb ball. After impact the velocities of the two balls are, respectively, 11.65 and 1.68 ft per sec to the left. Determine the velocity of the 100-lb ball before impact and also the value of e.

889. If in Fig. 584 the 20-lb ball is released from rest in the 60° position and swings and strikes the 30-lb ball, which is caused to swing through 45°, what is the coefficient of restitution for the two balls? Neglect the weights of the cords.

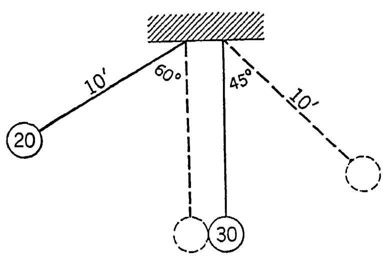

FIG. 584

191. Oblique Impact.—When two bodies collide in such a manner that their velocities before impact do not lie along the line connecting their centers of gravity, or if the surfaces of contact between the bodies are not perpendicular to the line connecting their centers of gravity, the impact is known as oblique impact.

If the surfaces of contact are smooth, the components of the velocities parallel to these surfaces remain unchanged during impact because there are no forces acting parallel to the surfaces. The components of the velocities normal to the surfaces are affected just as they are during direct central impact.

EXAMPLE

Assume that a 100-lb ball and an 80-lb ball, moving as indicated in Fig. 585 (*a*), meet in oblique impact. If $\dot{e}=0.6$, determine the amount and direction of the velocities after impact.

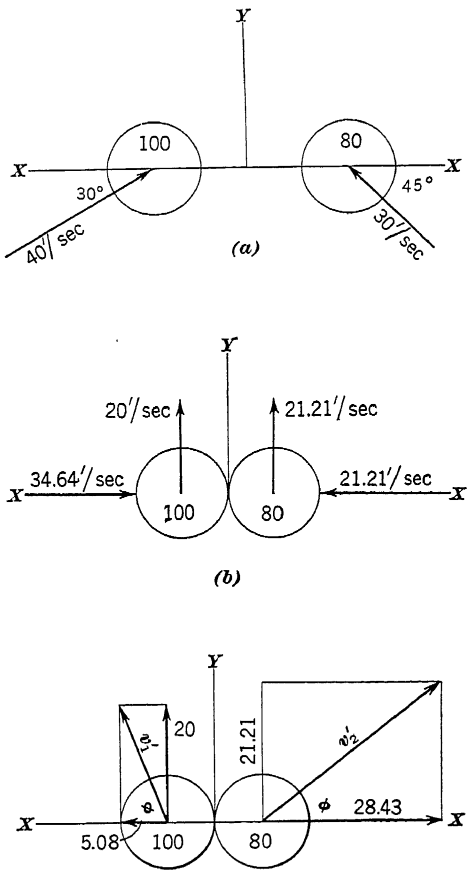

FIG. 585

In Fig. 585 (*b*) the two balls are shown during impact. The X axis is taken normal to the surfaces of contact and the Y axis is tangent to the surfaces of contact.

The velocities of the balls before impact are resolved into components parallel to the X and Y axes. Since the surfaces of contact are smooth, the components of the velocities parallel to the Y axis (20 and 21.21 ft per sec) are unchanged by the impact. We can apply equations $M_1 v_1 + M_2 v_2 = M_1 v_1' + M_2 v_2'$, and $(v_1 - v_2)\, e = (v_2' - v_1')$ to the X components.

$$\frac{100}{32.2} \times 34.64 + \frac{80}{32.2}(-21.21) = \frac{100}{32.2} v_{1x}' + \frac{80}{32.2} v_{2x}'$$

$$1{,}774 = 100\, v_{1x}' + 80\, v_{2x}' \qquad (1)$$

$$[34.64 - (-21.21)]\, 0.6 = v_{2x}' - v_{1x}'$$

$$33.51 = v_{2x}' - v_{1x}' \qquad (2)$$

Solving equations (1) and (2) gives

$v_{1x}' = -5.08$ ft per sec to left and $v_{2x}' = 28.43$ ft per sec to right

In Fig. 585 (*c*) the components and resultant velocities after impact are shown.

$$v_1' = \sqrt{5.08^2 + 20^2} = 20.6 \text{ ft per sec}$$

$$\tan \theta = \frac{20}{5.08} = 3.93$$

$$\theta = 75.75°$$

$$v_2' = \sqrt{28.43^2 + 21.21^2} = 35.5 \text{ ft per sec}$$

$$\tan \phi = \frac{21.21}{28.43} = 0.746$$

$$\phi = 36.7°$$

PROBLEMS

890. A ball is thrown so that it hits a smooth horizontal plane with a velocity of 80 ft per sec at an angle of 30° with the plane. If $e = 0.7$, what are the velocity and direction of the rebound? *Ans. 74.8 ft per sec; 22°.*

891. A 10-lb ball falls 20 ft and strikes a plane which is inclined 30° with the horizontal. If $e = 0.8$ for the plane and ball, what is the velocity of rebound? What is the loss in K.E.?

892. A 50-lb ball moving horizontally to the right with a velocity of 20 ft per sec is struck directly on top by a 10-lb ball which has fallen 20 ft. If $e = 0.5$, determine the velocities after impact.

893. A ball having a velocity of 60 ft per sec strikes a smooth surface at an angle of 45° with the surface. If it rebounds at an angle of 30° with the surface, what are the value of e and the velocity of rebound?

894. If $e=0.8$ for the two spheres shown in Fig. 586, determine the magnitudes and directions of the velocities after impact. *Ans.* $V_{40}=35.7$ *ft per sec; 98.5°;* $V_{30}=38.2$ *ft per sec; 344.8°.*

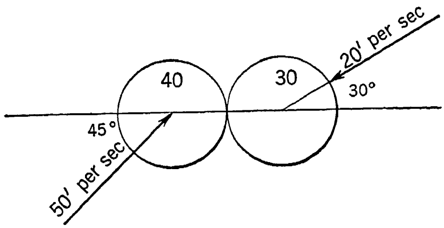

FIG. 586

192. Force Exerted by Jet of Water on a Smooth Deflecting Surface.

—*Case 1: A stationary flat plate perpendicular to the jet.*—When a jet of water strikes a stationary flat plate, Fig. 587, its velocity and momentum in the direction of the jet are reduced to zero. Let the mass of water striking the plate in 1 sec be the free body. From Art. 188

$$\text{Resultant Linear Impulse} = M(v_1 - v)$$

$$-P'\,t = \frac{w}{g}\,t\,a\,v\,(0-v)$$

where a is the cross-sectional area of the jet in square feet, v_1 and v are the velocities of the plate and jet in feet per second, and w is the weight of water in pounds per cubic foot. If t is taken as 1 sec,

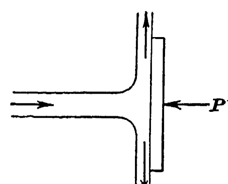

FIG. 587

$$P = P' = \frac{w}{g}\,a\,v^2 = \frac{W}{g}\,v$$

where W is the weight of water, in pounds per second, which strikes the plane; and $P=P'$ is the pressure of the water against the plate, in pounds.

Case 2: Flat plate moving.—If the plate in Fig. 587 is moving with a velocity v_1 in the direction of the jet, the velocity of the water relative to the plate is $(v-v_1)$.

The mass of water striking the plate in 1 sec is $\frac{w}{g}\,a\,t\,(v-v_1)$. It has its absolute velocity changed from v to v_1.

Resultant Linear Impulse = Change in Linear Momentum

$$-P'\,t=\frac{w}{g}\,a\,t\,(v-v_1)\,(v_1-v)$$

$$P'=\frac{W}{g}\,(v-v_1)$$

It is sometimes more convenient to express the change in momentum in terms of the velocity relative to the plate. As before, the mass of water striking the plate in 1 sec is $\frac{w}{g}\,a\,t\,(v-v_1)$ and it has its velocity relative to the plate changed from $v-v_1$ to zero in the direction of the jet. Therefore,

Resultant Linear Impulse = Change in Linear Momentum

$$-P'\,t=\frac{w}{g}\,a\,t\,(v-v_1)\,[0-(v-v_1)]$$

$$P=P'=\frac{W}{g}\,(v-v_1)$$

Case 3: *A curved vane moving with a constant velocity.*—The smooth vane, Fig. 588, is moving with a uniform velocity v_1 in the same direction as the velocity v of the jet. Since the vane is smooth, the water will leave the vane at C with the same relative velocity with which it entered at B, or $(v-v_1)$. The number of pounds of water entering the vane at B per second is $w\,a(v-v_1)$.

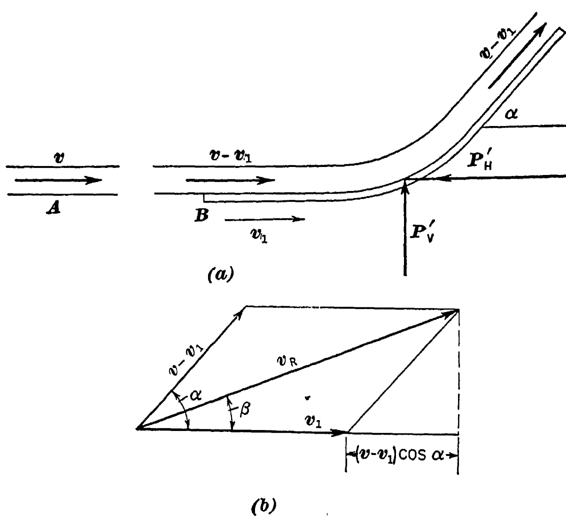

Fig. 588

When the water leaves the vane at C, its velocity has a component $(v-v_1)$ tangent to the vane and a component v_1 in the direction of the motion of the vane. The absolute, or resultant, velocity of the water at C is represented by v_R, Fig. 588 (*b*).

If the absolute velocities are considered, the change in momentum in the horizontal direction is given by the relation

$$-P'_H\, t = \frac{w}{g}\, a\, t\, (v-v_1) \Big\{[v_1+(v-v_1)\cos\alpha]-v\Big\}$$

If $t=1$,

$$P_H = P'_H = \frac{W}{g}(v-v_1)(1-\cos\alpha)$$

The change in momentum in the vertical direction is given by the relation

$$P'_V\, t = \frac{w}{g}\, a\, t\, (v-v_1)\,[(v-v_1)\sin\alpha - 0]$$

When $t=1$,

$$P_V = P'_V = \frac{W}{g}(v-v_1)\sin\alpha$$

If relative velocities are considered,

$$-P'_H\, t = \frac{w}{g}\, a\, t\, (v-v_1)\,[(v-v_1)\cos\alpha - (v-v_1)]$$

$$P_H = P'_H = \frac{W}{g}(v-v_1)(1-\cos\alpha)$$

$$P'_V\, t = \frac{w}{g}\, a\, t\, (v-v_1)\,[(v-v_1)\sin\alpha - 0]$$

$$P_V = P'_V = \frac{W}{g}(v-v_1)\sin\alpha$$

PROBLEMS

895. A jet from a nozzle 1 in. in diameter is directed against a flat plate. If the velocity of the jet is 25 ft per sec, what is the pressure on the plate? *Ans. 6.6 lb.*

896. Water under a head of 100 ft is discharged from a 1-in. nozzle. It strikes a flat plate which is moving away from the nozzle with a velocity of 20 ft per sec. What pressure does the water exert on the plate? Assume that $v=\sqrt{2\,g\,h}$.

897. Solve Problem 896 if the plate moves toward the nozzle with a velocity of 10 ft per sec.

898. A nozzle 2 in. in diameter discharges water at a velocity of 40 ft per sec against a stationary curved vane. If the vane turns the water 60° away from the direction of the jet, what are the horizontal and vertical components of the pressure against the vane?

899. If the vane in Problem 898 is moving in the same direction as the jet with a velocity of 10 ft per sec, what are the horizontal and vertical components of the pressure against the vane?

900. A curved vane turns water from a 1-in. diameter jet through 150°. If the head on the jet is 200 ft, what pressure will the jet exert against the vane in the direction of the jet? *Ans. 254 lb.*

193. Moment of Momentum and Angular Momentum.—In Fig. 589, M represents any body which, at any given instant, has an angular velocity ω about the axis through O, normal to the plane of the paper, due to the action of the external forces F_1 and F_2. If dm is any particle of mass at a distance ρ from the axis through O, then the momentum of dm is $dm\ v = dm\ \rho\ \omega$. The momentum $dm\ \rho\ \omega$ is a vector quantity and it has a moment with respect to the axis through O which is $\rho\ dm\ \rho\ \omega$.

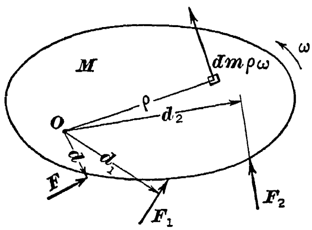

Fig. 589

Each and every particle dm of the mass M has a similar moment of momentum with respect to the axis through O. Integration over the entire mass then gives

$$\text{Moment of Momentum} = \int \rho^2\,\omega\,dm = I_0\,\omega$$

where I_0 is the moment of inertia of the entire mass with respect to the axis through O. This expression is also called angular momentum.

194. Relation of Angular Impulse to Angular Momentum. If force F acting at a distance d from O in Fig. 589 is the resultant of all the external forces F_1, F_2, etc. which act on the mass M, then by Art. 152

$$\text{Resultant Torque} = F\,d = I_0\,\alpha$$

Since $\alpha = \dfrac{d\omega}{dt}$,

$$\int F\,d\,dt = \int I_0\,\frac{d\omega}{dt}\,dt = \int_{\omega_0}^{\omega} I_0\,d\omega$$

$$F\,t\,d = I_0\,\omega - I_0\,\omega_0$$

The term $F\,t\,d$ is the moment of the resultant impulse, or the resultant angular impulse which acts during time t. The equation can then be written:

Resultant Angular Impulse = Change in Angular Momentum

The equation can also be transformed into an equation which is similar to the equation given for linear impulse and momentum in Art. 188. Thus,

$$\text{I.A.M.} + \text{P.A.I.} - \text{N.A.I.} = \text{F.A.M.}$$

The dimensions of angular impulse are

$$F\,t\,d = \text{lb} \times \text{ft} \times \text{sec} = \text{ft-lb-sec}$$

Those for angular momentum are

$$\int \rho^2\,\omega\,dm = \text{ft}^2 \times \frac{1}{\text{sec}} \times \frac{\text{lb}}{\text{ft}} \times \text{sec}^2 = \text{ft-lb-sec}$$

An angular impulse or an angular momentum may be represented graphically by a vector drawn parallel to its axis of rotation. The sense of the vector is determined by the right-hand screw rule. If the observed rotation is clockwise, the vector points away from the observer; if the rotation is counter-clockwise, the vector points toward the observer.

Since angular impulse and angular momentum are vector quantities, they may be resolved into components or may be combined into resultants, as is done with force vectors.

When the equation just given is applied to any free body for which the positive angular impulse (P.A.I.) is equal to the negative angular impulse (N.A.I.), then the initial angular momentum (I.A.M.) is equal to the final angular momentum (F.A.M.). Such a situation occurs when two rotating masses M_1 and M_2 interact, as when they are suddenly joined by a clutch or similar device. The angular impulses are then equal and oppositely directed and the equation becomes

$$I_1\,\omega_1 + I_2\,\omega_2 = (I_1 + I_2)\,\omega \tag{1}$$

where ω_1 and ω_2 are the angular velocities before impact and ω is the common angular velocity after impact.

If the objects are partially elastic and are free to separate after impact, the equation becomes

$$I_1\,\omega_1 + I_2\,\omega_2 = I_1\,\omega_1' + I_2\,\omega_2' \tag{2}$$

where ω_1' and ω_2' are the angular velocities after impact occurs.

If r is the radius at which the two equal and opposite impulses act, it follows from Art. 190 that

$$(v_1 - v_2)\, e = v_2' - v_1'$$

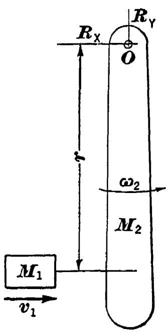

FIG. 590

Since $v = r\,\omega$,

$$(r\,\omega_1 - r\,\omega_2)\, e = r\,\omega_2' - r\,\omega_1'$$

or

$$(\omega_1 - \omega_2)\, e = \omega_2' - \omega_1' \tag{3}$$

Equations (2) and (3) can be solved for the final angular velocities.

Equations (2) and (3) can also be applied to the case where M_1 is the mass of a translating body and M_2 is the mass of a rotating body, as in Fig. 590.

The initial linear momentum of M_1 is $M_1\,v_1$, and the moment of this momentum with respect to the supporting axis through O is $M_1\,v_1\,r$. The linear momentum of this mass after impact is $M_1\,v_1'$, and the angular momentum is $M_1\,v_1'\,r$. If I_{O_2} is the moment of inertia of mass M_2 with respect to the axis through O, then equations (2) and (3) may be written as follows:

$$M_1\,v_1\,r + I_{O_2}\,\omega_2 = M_1\,v_1'\,r + I_{O_2}\,\omega_2' \tag{4}$$

$$(v_1 - r\,\omega_2)\, e = (r\,\omega_2' - v_1') \tag{5}$$

EXAMPLE 1

Assume a solid disk A, 1 ft in diameter and weighing 100 lb, which is free to turn on a shaft B. A second disk C, 3 ft in diameter and weighing 400 lb, is keyed to shaft B. If the disk A is caused to rotate 360 rpm and then suddenly is connected to disk C by throwing in a clutch, what will be the angular velocity, in rpm, of the two disks when rotating together? Neglect weight of shaft.

In this case the mutual angular impulses of the two disks on each other are equal and opposite and can be disregarded. Therefore,

Initial angular momentum = Final angular momentum

$$\frac{1}{2}\times\frac{100}{32.2}\times 0.5^2\times 12\pi = \left(\frac{1}{2}\times\frac{100}{32.2}\times 0.5^2 + \frac{1}{2}\times\frac{400}{32.2}\times 1.5^2\right)\omega$$

$$\omega = 1.02 \text{ rad. per sec or } \frac{1.02\times 60}{2\pi} = 9.74 \text{ rpm}$$

EXAMPLE 2

A 500-lb cylinder 3 ft in diameter is turning 600 rpm. A brake-shoe is pressed against the circumference with a normal pressure of 120 lb. How long will the cylinder continue to rotate if $f=0.4$ for the brake-shoe?

$$\text{I.A.M.}+\text{P.A.I.}-\text{N.A.I.}=\text{F.A.M.}$$

$$\frac{1}{2}\times\frac{500}{32.2}\times 1.5^2\,(20\pi)+0-120\times 0.4\times 1.5\,t=0$$

$$t=15.2 \text{ sec}$$

PROBLEMS

901. A 200-lb disk 2 ft in diameter, turning 120 rpm, and a 500-lb disk 5 ft in diameter, turning 900 rpm, are both turning freely in the same direction on the same shaft. The disks are suddenly joined together by a clutch. What is the angular velocity, in rpm, of the two disks after the clutch goes into action? *Ans. 853 rpm.*

902. If in Problem 901 the clutch is replaced by rigid projections so that after impact the disks are free to separate, and $e=0.9$, what angular velocities in rpm will the disks have after impact?

903. Determine the torque which is necessary to increase the speed of a 1,000-lb cylinder, 6 ft in diameter, from 120 rpm to 180 rpm in 20 sec.

904. A 10-lb sphere, 1 ft in diameter, is rotating 300 rpm in a horizontal plane at the end of a wire 6 in. long. If the length of the wire is gradually increased to 18 in., what is the angular velocity of the sphere in rpm?

905. If $f=0.4$ for the brake in Fig. 591, and the drum is turning 120 rpm when the brake is applied, determine the pressure P required to bring the drum to rest in 45 sec. (Cut cord and write two equations.)

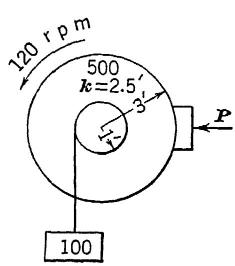

FIG. 591

906. In Fig. 590 a 96.6-lb symmetrical slender rod 5 ft long is supported on a frictionless pin at a point O which is 1 ft in from the end. It is struck at a point 3 ft below the pin at O by a 32.2-lb mass moving horizontally to the right with a velocity of 20 ft per sec. If $e=0.8$, how high will the center of gravity of the rod swing? *Ans. 1.62 ft.*

195. Solution of the Motion of a Rigid Body by the Principles of Impulse and Momentum.—Since both impulse and momentum are vector quantities, the equations developed in Arts. 188 and 194 can often be conveniently applied to problems involving translation, rotation, or plane motion of a rigid body.

Many of the problems of Chapters 15, 17, 18, and 19 can be advantageously worked by applying these equations. However, it will first be necessary to examine a rigid body in plane motion to determine the effect of this motion on the angular momentum of the body.

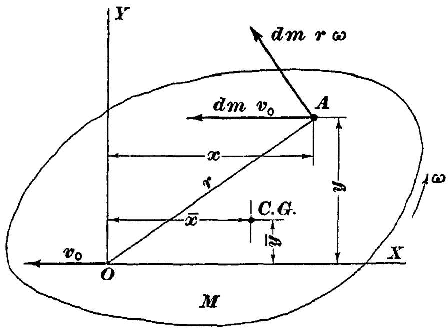

FIG. 592

Let Fig. 592 represent any mass M which is moving with plane motion in the plane XOY. At a given instant it is rotating with an angular velocity ω about an axis through O perpendicular to the plane XOY. The axis through O has a linear velocity v_O directed horizontally to the left along the X axis. If dm is the mass of any particle at point A at a distance r from the axis through O, the absolute velocity of the particle at A is

$$v_A = v_O \not\rightarrow v_{\frac{A}{O}} = v_O \not\rightarrow r\,\omega$$

The absolute linear momentum of the mass dm is $dm\,v_A = dm\,(v_O + r\,\omega)$. Therefore, the angular momentum of the entire mass M about the axis through O is

$$\begin{aligned}\text{Resultant angular momentum} &= \int dm\, v_O\, y + \int dm\, r\,\omega\, r \\ &= v_O \int y\, dm + \omega \int r^2\, dm \\ &= M\,\bar{y}\, v_O + I_O\,\omega\end{aligned}$$

Examination of this equation reveals that the resultant momentum will reduce to $I_O\,\omega$ when

(*a*) the reference point O is at the center of gravity C.G. because then $\bar{y}=0$;

(*b*) the velocity v_O of the reference point O is 0;

(*c*) the velocity v_O of the reference point O is directed through the C.G. because then $\bar{y}=0$.

Generally the C.G. is the most convenient reference point to employ. When the reference point O for any plane motion coincides with the C.G.,

$$\text{Resultant angular momentum} = I_O\,\omega$$

where I_O is the moment of inertia of the mass with respect to an axis through the C.G. normal to the plane of the paper and ω is the angular velocity of the mass M with respect to that axis.

If the linear impulses and momentums are resolved into their components along any two convenient axes through the C.G., the equation of Art. 188 can be applied first to the components acting parallel to one of these axes and then to the components which are parallel to the second axis. For either group of components,

$$\text{I.L.M.} + \text{P.L.I.} - \text{N.L.I.} = \text{F.L.M.}$$

Two independent equations are thus obtained.

Similarly, the equation of Art. 194 can be applied to give the relationship between the angular impulses and the angular momentums with respect to an axis through the C.G. The three equations can then be solved simultaneously.

EXAMPLE 1

Solve Example 2, Art. 144, by applying the equations of Arts. 188 and 194.

Using Fig. 436 (*b*) as a free body after removing the inertia force, apply the linear impulse-momentum equation

$$\text{I.L.M.} + \text{P.L.I.} - \text{N.L.I.} = \text{F.L.M.}$$

$$\text{Average velocity} = \frac{50}{5} = 10 \text{ ft per sec}$$

$$\text{Final velocity} = 10 \times 2 = 20 \text{ ft per sec}$$

$$5T - 100 \times 5 - 866 \times 5 = \frac{1{,}000}{32.2} \times 20$$

$$T = 1{,}090.2 \text{ lb}$$

Apply the same equation to Fig. 436 (*c*) after removing the inertia force.

$$5W - 2 \times 1{,}090.2 \times 5 = \frac{W}{32.2} \times 10$$

$$W = 2{,}324 \text{ lb}$$

EXAMPLE 2

If any solid sphere weighing W lb rolls freely down a plane for which the limiting value of the static coefficient of friction is f, what is the maximum allowable value of the angle θ, Fig. 593?

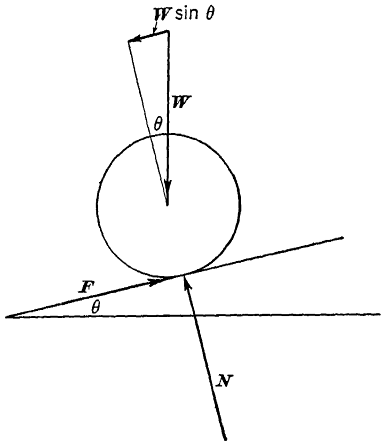

FIG. 593

Apply the linear impulse-momentum equation to the components parallel to the plane.

$$\text{I.L.M.}+\text{P.L.I.}-\text{N.L.I.}=\text{F.L.M.}$$

$$0+W\,t\sin\theta-F\,t=\frac{W\,v}{g}=\frac{W\,r\,\omega}{g} \tag{1}$$

Apply the same equation to the components normal to the plane.

$$0+N\,t-W\,t\cos\theta=0 \tag{2}$$

Apply the general angular impulse-momentum equation with respect to an axis through the center of the sphere.

$$\text{I.A.M.}+\text{P.A.I.}-\text{N.A.I.}=\text{F.A.M.}$$

$$0+F\,r\,t-0=\frac{2}{5}\,\frac{W\,r^2\,\omega}{g} \tag{3}$$

If $\dfrac{W\,r\,\omega}{g}$ is eliminated from equations (1) and (3), the following equation is obtained.

$$W \sin \theta - \frac{7}{2}F = 0$$

Since $F = f\,W \cos \theta$,

$$W \sin \theta - \frac{7}{2} f\,W \cos \theta = 0$$

$$\tan \theta = \frac{7}{2} f$$

EXAMPLE 3

Determine the linear velocity of the 300-lb weight in Example 1, Art. 155, at an instant 10 sec after it starts from rest.

Use Fig. 480 (*c*) as the first free body after removing the inertia force.

$$\text{I.L.M.} + \text{P.L.I.} - \text{N.L.I.} = \text{F.L.M.}$$

$$0 + 300 \times 10 - T \times 10 = \frac{300}{32.2}\,v \tag{1}$$

Use Fig. 480 (*b*) as a free body after removing the inertia force.

$$\text{I.A.M.} + \text{P.A.I.} - \text{N.A.I.} = \text{F.A.M.}$$

$$T \times 3 \times 10 - 200 \times 0.4 \times 1 \times 10 = \frac{500}{32.2} \times 2.5^2 \times \frac{v}{3} \tag{2}$$

Solve equations (1) and (2) for v.

$$v = 136.1 \text{ ft per sec}$$

PROBLEMS

907. Solve Problem 615, Fig. 451, by the method illustrated in Art. 195.

908. What is the linear velocity of the cylinder in Problem 835 and Fig. 558 at an instant 10 sec after it starts from rest? Use the method illustrated in Art. 195. *Ans. 42.9 ft per sec.*

909. Solve Problem 838 by the equations given in Art. 195. What is the linear velocity of the reel after moving for 10 sec from rest?

910. Solve Problem 678, Fig. 483, for the braking force P. Use the impulse-momentum method.

911. Solve Example 2, Art. 184, by the impulse-momentum method.

912. Solve Problem 841 by the impulse-momentum method.

196. Gyroscope.—When a body with a relatively large rotative speed or moment of inertia rotates about its axis of sym-

metry, it resists any attempt to change the direction of its axis of rotation. This phenomenon is known as the gyroscopic effect. It is caused to serve a useful purpose in the gyroscopic compass, in gyroscopic stabilizers, and in gyroscopic steering mechanisms. The discussion which follows is limited to the cases in which the X, Y, and Z axes are mutually rectangular.

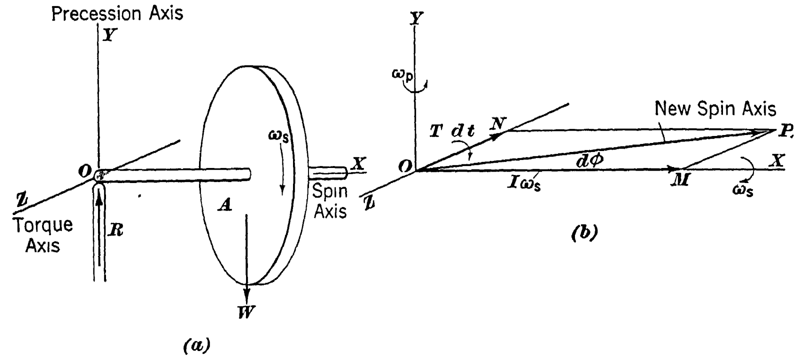

FIG. 594

In Fig. 594 (a) the disk A is rotating clockwise with an angular velocity of ω_s rad. per sec about the axis OX, or the spin axis. Its angular momentum is $I\ \omega_s$ where I is the moment of inertia with respect to the spin axis OX. This momentum is represented graphically (see Art. 194) in Fig. 594 (b) by the vector OM. The spin axis OX is free to turn in any direction about pivot O. If the disk were not rotating, the axis OX would turn about the axis OZ, or the torque axis, because of the clockwise torque T due to the weight W and the reaction R along axis OY, or the precession axis. However, when the disk is rotating about the spin axis OX, the spin axis rotates about the precession axis OY instead of about the torque axis OZ.

The couple consisting of the weight W and the reaction R applies a clockwise torque T about the torque axis OZ. During any time dt this torque supplies an angular impulse $T\ dt$, which is equal to the change in momentum occurring in time dt with respect to the torque axis OZ. The resultant angular momentum of the disk is therefore the vector $I\ \omega_s \nrightarrow T\ dt$, shown graphically in Fig. 594 (b) as the vector OP. If OP is the resultant momentum, it is also the new spin axis; and the original spin axis OX of the disk must precess about the precession axis OY to the position

OP. As long as ω_s and the torque T are maintained, the precession about axis OY will continue.

From Fig. 594 (*b*),

$$ON = OM \tan d\phi$$

But $d\phi$ is so small that $\tan d\phi = d\phi$. Hence,

$$T\,dt = I\,\omega_s\,d\phi$$

or

$$T = I\,\omega_s\,\frac{d\phi}{dt}$$

Since $\dfrac{d\phi}{dt} = \omega_p$, which is the angular velocity of precession with respect to axis OY,

$$T = I\,\omega_s\,\omega_p$$

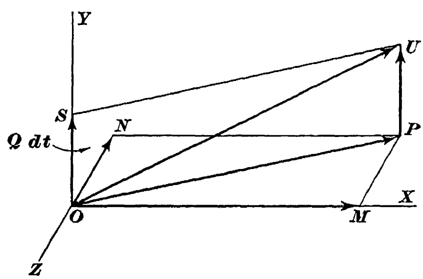

Fig. 595

If any couple Q, Fig. 595, acts in the XZ plane to increase the velocity of the counter-clockwise precession with respect to axis OY, the rotating disk and its shaft OP will rise. The newly applied impulse $Q\,dt$ produces an equal change in the momentum about the axis OY, which is represented by the vector OS in Fig. 595. When this vector is combined with OP, the resultant vector is OU, which is the new position of the spin axis. If an attempt is made to retard the precession about the axis OY, the vector OS will be reversed and the resultant vector OU will fall below the XZ plane; or the disk and its axis OP will fall.

If a torque with respect to an axis perpendicular to the spin axis causes a precession about a third axis perpendicular to the plane containing the first two axes, it is reasonable to expect that a forced precession about an axis perpendicular to the spin axis will cause an induced torque with respect to an axis perpendicular to the plane of the spin axis and the precession axis. This

occurs when a railroad car or an automobile goes around a curve. The forced precession of the car wheels causes an increased wheel reaction at the outer wheel and a decreased reaction at the inner wheel. These changes are independent of the wheel reactions due to gravity and inertia forces.

EXAMPLE

A car goes around a horizontal curve of 250-ft radius at 30 mi per hr. Each wheel weighs 70 lb, is 30 in. in diameter, and has a radius of gyration of 12 in. Determine the change in wheel pressure caused by the gyroscopic effect if the distance between wheel centers (tread) is 5 ft.

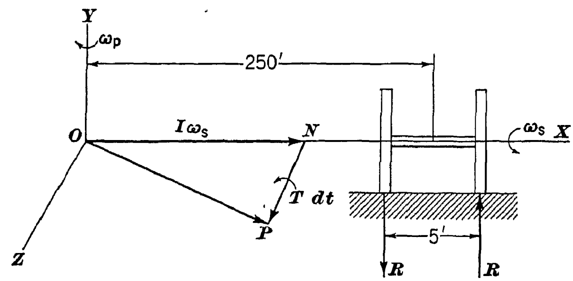

FIG. 596

The moment of inertia of one pair of wheels is

$$I=\frac{W}{g}k^2=\frac{2\times 70}{32.2}\left(\frac{12}{12}\right)^2=4.34$$

$$\omega_s=\frac{44}{1.25}=35.2 \text{ rad. per sec}$$

$$\omega_p=\frac{44}{250}=0.176 \text{ rad. per sec}$$

Assuming that the center of curvature is at O, Fig. 596, and the car is going around the curve in a clockwise direction when observed from above (approaches the observer), the forced precession is about the axis OY, as indicated by vector ω_p. The vector representing the angular momentum with respect to the spin axis is ON, it precesses to OP, and vector NP represents the change in angular momentum caused by the induced torque acting at the wheel reactions. If the right-hand screw rule is applied to vector NP, it will be observed that the direction of the

induced torque caused by the gyroscopic effect is counter-clockwise, and the reaction on the outside wheel is increased.

$$\text{Torque} = R \times 5 = I\ \omega_s\ \omega_p = 4.34 \times 35.2 \times 0.176$$
$$R = 5.38 \text{ lb}$$

PROBLEMS

913. A disk like that in Fig. 594 is 12 in. in diameter and weighs 64.4 lb, and it rotates with an angular velocity of 1,200 rpm in a direction opposite to that indicated in Fig. 594. The distance between the disk and the frictionless unrestricted pivot at O is 12 in. Determine the direction and the angular velocity of precession. *Ans. 2.05 rad. per sec; clockwise, viewed from above.*

914. The solid disk of a gyroscope is 2 ft in diameter and weighs 96.6 lb. It is mounted on a horizontal shaft which is supported by a pivot 2 ft from the disk. The shaft turns in a horizontal plane with an angular velocity of 6 rpm with respect to the pivot. Determine the angular velocity of the disk, in rpm.

915. An airplane approaches a landing field at a steep angle and changes to a path parallel to the ground. The propeller is turning clockwise when observed from the rear. How will the gyroscopic effect of the propeller change the direction of the plane? *Ans. Plane will turn to right, observed from above.*

916. An airplane propeller, 10 ft in diameter and weighing 180 lb, turns 2,500 rpm when the plane is traveling 250 mi per hr. If $k = 3$ ft, what is the gyroscopic torque when the plane makes a horizontal turn of 800-ft radius?

917. A steam-turbine rotor, which weighs 12,880 lb, rotates at 1,800 rpm with its shaft parallel to the longitudinal axis of the ship it drives. The turbine bearings are 6 ft apart and $k = 2.5$ ft. If the ship turns on a curve of 2,000-ft radius when its speed is 30 knots, what is the change in bearing pressure? 1 knot = 6,080.26 ft per hr. *Ans. 1,986 lb.*

REVIEW PROBLEMS

918. A 100-lb sand bag starts from rest and slides down a 30° plane for 10 sec; it then strikes a 300-lb sand bag which is moving down the plane with a velocity of 5 ft per sec. If $f = 0.4$ for the plane, what are the velocities of the bags after impact? *Ans. 16.1 ft per sec.*

919. A 100,000-lb car has a speed of 10 mi per hr at the top of a 5% grade. The car travels down the 5% grade for ½ min and then goes up an 8% grade. Car resistance is 10 lb per ton. When will the car come to rest?

920. The reciprocating table of a planing machine and its load weigh 12,000 lb and are moved back and forth in a horizontal direction by a 300-lb horizontal driving force. The horizontal frictional resistance is 100 lb. How long does it take to change the table speed from 36 ft per min to 90 ft per min in the opposite direction? *Ans. 3.35 sec.*

921. A gun weighing 80,000 lb gives a 200-lb shell a muzzle velocity of 1,600 ft per sec. The recoil of the gun is resisted by a constant force of 15,000 lb. Determine the time during which the gun is in motion and the distance moved by it.

922. A 500-lb projectile is given a muzzle velocity of 2,000 ft per sec by a 120,000-lb gun. The gun recoils 3 ft against a nest of springs. What is the scale of the springs, if the mass of the explosive gases is neglected?

923. Solve Problem 922 if the weight of the powder charge is 350 lb and its velocity is 1,000 ft per sec.

924. A 10-lb projectile is shot into a 640-lb sand bag, which is at rest on a horizontal plane for which $f = 0.5$. If the bag containing the projectile moves 15 ft after impact, what was the velocity of the projectile? *Ans. 1,428 ft per sec.*

925. If in Problem 924 the sand bag is at rest on a 30° plane for which $f = 0.3$ and the striking velocity of the projectile parallel to the plane is 1,600 ft per sec (up), how long after impact will the bag and projectile continue to move? *Ans. 1.004 sec.*

926. Two hand-cars weighing 200 lb and 350 lb are standing close together on the same track. If a 180-lb man jumps with a velocity of 10 ft per sec from the 200-lb car to the 350-lb car, determine the velocity of each car after the man jumps.

927. A $\frac{1}{2}$-lb bullet is fired into a 200-lb box of sand, which is at rest on a horizontal plane. The box moves 2 ft. If $f = 0.3$ for the plane, what was the velocity of the bullet? *Ans. 2,494 ft per sec.*

928. A 3-lb hammer moving with a velocity of 30 ft per sec strikes a 1-oz nail. If the nail is driven $\frac{1}{2}$ in., what is the average resistance offered by the wood? What per cent of the energy of the hammer was wasted?

929. If the average resistance offered by the ground to an 800-lb pile is 60,000 lb, how far must a 600-lb driver fall to drive the pile 1 in. at each stroke? The impact losses are neglected and the driver remains in contact with the pile.

930. A 1-lb bullet with a velocity of 1,600 ft per sec is shot into a sand bag which is suspended from the end of a 4-ft rope. If the bag swings so that the rope makes a 30° angle with the vertical, what was the weight of the sand bag? *Ans. 271 lb.*

931. An 80-lb weight, moving with a velocity of 10 ft per sec, strikes a 50-lb body moving in the opposite direction with a velocity of 8 ft per sec. If the coefficient of restitution is 0.5, what are the velocities of the bodies after impact?

932. A 5-lb ball, falling vertically with a velocity of 20 ft per sec, is struck on the side by a smooth 8-lb ball moving horizontally with a velocity of 10 ft per sec. If $e = 0.6$, determine the velocities and directions of the balls after impact.

933. A 20-lb ball moves horizontally to the right with a velocity of 10 ft per sec. A 10-lb ball moves horizontally to the left with a velocity of 12 ft per sec. When impact occurs, a line connecting the centers of gravity of the balls makes an angle of 30° with the horizontal. Determine the amounts and directions of the velocities after impact, if $e = 0.6$.

934. From a point 5 ft above the floor a ball is thrown horizontally and normally toward a smooth wall 25 ft away. If $e = 0.6$ for the wall and ball, and the initial velocity of the ball is 75 ft per sec, when and where will the ball strike the ground?

935. Fig. 597 is the projection on a horizontal plane of the path of a ball thrown horizontally with a velocity of 50 ft per sec from point A against a vertical wall at B and striking the floor at C, which is 6 ft lower than A. If the coefficient of restitution is $e=0.8$, determine the distances d and x and also the striking velocity at C.

936. A 3-in. diameter jet of water flowing under a head of 150 ft strikes a flat vane which is moving with a velocity of 30 ft per sec in the same direction as the water. How many horsepower can the vane produce?

937. Solve Problem 936 if the vane is curved through 150°.

938. In Problem 936 what speed of the flat vane would produce the maximum horsepower?

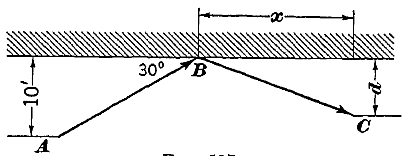

FIG. 597

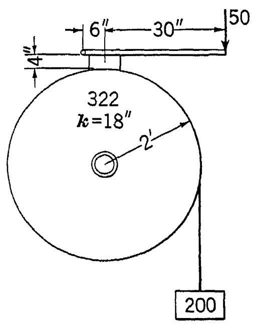

FIG. 598

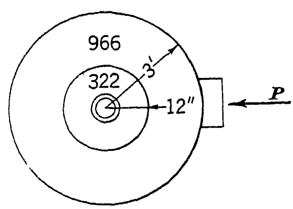

FIG. 599

939. A jet of water flows from a horizontal nozzle at the rate of 20 lb per sec under a head of 350 ft. It enters a frictionless curved blade which turns the water through 120°. The blade has a velocity of 50 ft per sec in the same direction as the jet. Determine the horsepower developed by the blade and the side thrust which must be carried by the thrust bearing.

940. A 75-lb machine gun is mounted on a hand-car which weighs 300 lb. The gun is pointed in a horizontal direction parallel to the tracks on which the car runs. If the gun delivers 300 bullets per minute for 20 sec and each bullet weighs 1 oz, how far will the car move? The frictional resistance of the car is 15 lb, and the bullet velocity is 2,000 ft per sec.

941. If the car in Problem 940 is 10 ft long and is assumed to be frictionless, and a 160-lb man walks from one end to the other, how far will the car move? *Ans. 2.58 ft.*

942. Determine the velocity of the 200-lb weight in Fig. 598 10 sec after it starts from rest, if $f=0.4$ for the brake.

943. In Fig. 599 a 322-lb cylinder 2 ft in diameter, turning 240 rpm, and a 966-lb cylinder 6 ft in diameter, turning 120 rpm, are rotating counter-

clockwise on the same shaft. They are suddenly locked together by a clutch, and a brake is applied with a normal pressure P. If $f=0.5$ for the brake, determine the force P required to stop the cylinders in 30 sec. *Ans. 40.4 lb.*

944. A 0.25-lb bullet, which has a velocity of 1,500 ft per sec in the horizontal direction, embeds itself in the 96.6-lb timber shown in Fig. 600, which is supported at A on a frictionless pin. The bullet strikes the timber so that it produces no change in the reaction at A. What is the angular velocity of the timber after impact? *Ans. 0.86 rad. per sec.*

945. Fig. 601 represents a mine-hoist cable drum and brake drum. The weight of the drums is 3,000 lb, $k=3$ ft, and $f=0.6$ for the brake band. If the drums are turning 120 rpm when the brake is applied, what force P will bring the 6,000-lb car to rest in 20 sec? What is the tension in the cable while the car is coming to rest?

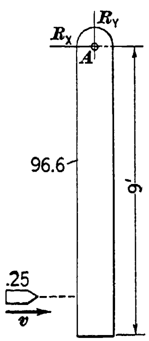

FIG. 600

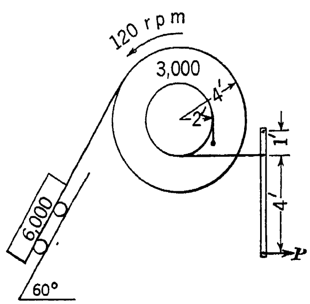

FIG. 601

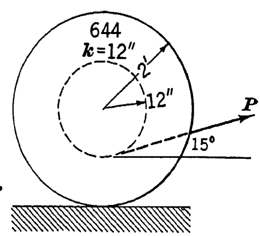

FIG. 602

946. If the car in Problem 945 is going up the incline with a speed of 40 ft per sec when the power is shut off and $P=0$, how long will the car continue to ascend? *Ans. 1.836 sec.*

947. A solid cylinder 2 ft in diameter and weighing 193.2 lb starts from rest and rolls down a 30° plane. Determine the rpm after 10 sec. Also compute the coefficient of friction, if the cylinder is just about to slip.

948. Determine the time required for the center of gravity of a 64.4-lb solid sphere 2 ft in diameter to attain a linear velocity of 50 ft per sec after starting from rest and rolling down a 30° plane without slipping. What is the required frictional force?

949. Determine the force P required to give the spool in Fig. 602 a speed of 25 rpm 5 sec after starting from rest, if there is no slipping.

950. Determine the time required for the 96.6-lb weight in Fig. 603 to attain a velocity of 10 ft per sec after starting from rest. The spool rolls along the 15° incline without slipping.

951. Solve for the angular velocity, in rpm, of the spool in Fig. 603 10 sec after it starts from rest if the cord comes off the spool at a point 180 ft from that shown.

952. A 10-oz bullet with a velocity of 1,000 ft per sec is shot into a 64.4-lb solid sphere 1 ft in diameter, which is at rest on a 30° plane. The

bullet travels parallel to the plane and strikes the sphere with direct central impact. Assume that the bullet remains at the center of the sphere, and neglect any variation in density. For what time and distance will the sphere roll up the plane? *Ans. 0.598 sec; 2.06 ft.*

953. A pair of locomotive driving wheels 7 ft in diameter and their connecting axle weigh 7,000 lb. If $k = 3$ ft, what is the gyroscopic torque when the locomotive goes around a curve of 2,500-ft radius at 60 mi per hr? Determine also the resultant upward reaction at each wheel caused by gravity and the gyroscopic torque if the wheels are 4.9 ft center to center.

954. If an airplane motor turns clockwise when observed from the rear, what gyroscopic effect is produced when the plane makes an inside vertical loop?

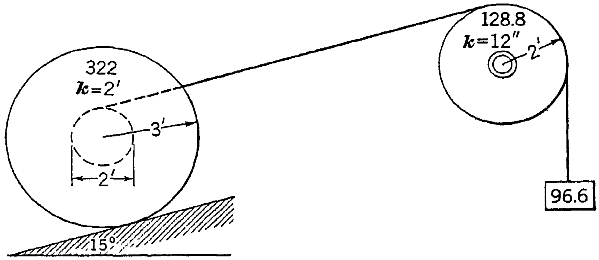

FIG. 603

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